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On the diameter of separated point sets with many nearly equal distances☆

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Abstract

A point set is *separated* if the minimum distance between its elements is 1. We call two real numbers *nearly equal* if they differ by at most 1. We prove that for any dimension $d \geq 2$ and any $\gamma > 0$, if P is a separated set of n points in $\mathbf{R}^d$ such that at least γn^2 pairs in $\binom{P}{2}$ determine nearly equal distances, then the diameter of P is at least $C(d, \gamma)n^{2/(d-1)}$ for some constant $C(d, \gamma) > 0$. In the case of $d = 3$, this result confirms a conjecture of Erdős. The order of magnitude of the above bound cannot be improved for any d.

1. Introduction

Erdős asked and partially answered numerous questions on the distribution of distances among n points in a Euclidean space [10,17,20,16,14,7]. Perhaps the best known question of this type is the so-called *"unit distance problem"* he raised in 1946 [9]: Given n points in the plane (or, more generally, in $\mathbf{R}^d$), at most how many of the $\binom{n}{2}$ interpoint distances can coincide? It is conjectured that in the plane this maximum is $n^{1+\frac{\text{const}}{\log \log n}}$, which is asymptotically sharp, for example for a $\sqrt{n} \times \sqrt{n}$ piece of the integer lattice. The best known upper estimate is only $O(n^{4/3})$ [21,

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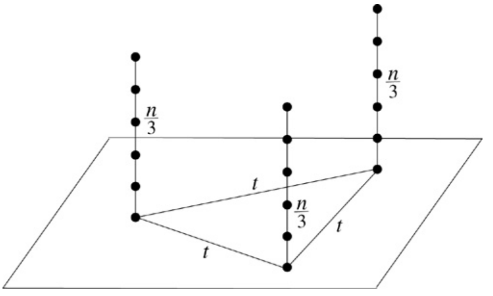

Fig. 1. n points in $\mathbf{R}^3$ can determine $\frac{1}{3}n^2$ nearly equal distances.

22]. In 3-space, the currently best upper bound is $n^{3/2}\beta(n)$, where $\beta(n)$ is an extremely slowly increasing function related to the inverse Ackermann function [8]. However, the true order of magnitude of this function is probably closer to $n^{4/3}$. In higher dimensions, the asymptotically tight answers are (see, e.g., [17]):

$$n^2\left(\frac{1}{2}-\frac{1}{d}\right)+O(n) \quad \text{if } d \geq 4 \text{ is even,}$$

$$n^2\left(\frac{1}{2}-\frac{1}{d-1}\right)+O(n^{4/3}) \quad \text{if } d \geq 5 \text{ is odd.}$$

The exact answer is known if $d = 4$ [6]. These questions are intimately related to problems concerning incidences between points and curves, surfaces, etc. (See [2,19].)

Erdős observed that the answer to the unit distance problem does not remain the same if one counts the number of distances that are *nearly equal*, where several distances are said to be nearly equal if they differ by at most 1, i.e. they all lie in an interval $[t, t+1]$ for some $t > 0$. To exclude trivial examples, we consider only *separated* point sets, i.e. point sets in which the minimum distance between two points is at least 1. Erdős et al. [11,12] proved that for any $t > 0$, $d \geq 2$, and for any separated set P of n points (vectors) in $\mathbf{R}^d$, the number of point pairs $\{\mathbf{u}, \mathbf{v}\} \subset P$ with $\|\mathbf{u}-\mathbf{v}\| \in [t, t+1]$ is at most $T(d, n) = \frac{n^2}{2}(1-\frac{1}{d}+o(1))$, as n tends to infinity. Here, $T(d, n)$ denotes the number of edges in a *balanced d-partite complete graph* on n vertices [3], i.e. in a graph whose vertices are divided into d classes, each having $\lfloor \frac{n}{d} \rfloor$ or $\lceil \frac{n}{d} \rceil$ elements, and two vertices are connected by an edge if and only if they belong to different classes. This is known to be the maximum number of edges that a K_{d+1}-free graph of n vertices can have.

The above upper bound on the number of point pairs $\{\mathbf{u}, \mathbf{v}\} \subset P$ with $\|\mathbf{u}-\mathbf{v}\| \in [t, t+1]$ can be attained for every $t \geq t(d, n)$, as shown by the following construction. Let $w_1, w_2, \ldots, w_d$ be the vertices of a regular $(d-1)$-dimensional simplex of edge length t, lying in the hyperplane $x_d = 0$. At each w_i, draw a line perpendicular to the hyperplane $x_d = 0$, and on each of these lines pick $\lfloor n/d \rfloor$ or $\lceil n/d \rceil$ distinct points whose x_d-coordinates are integers between 0 and n/d, so that the total number of points is n (see Fig. 1 for $d = 3$). If t is sufficiently large depending on d and n (roughly $\frac{1}{2d^2}n^2$), the distance between any pair of points selected on different perpendicular lines belongs to the interval $[t, t+1]$.

The question arises, what is the minimal diameter of a separated set of n points in $\mathbf{R}^d$ with $\Omega(n^2)$ nearly equal distances? It is not hard to see (using the Pythagorean theorem) that the answer in the plane is $\Theta(n^2)$. The problem becomes more interesting in higher dimensions. Notice that the diameter of the 3-dimensional configuration depicted in Fig. 1 is $\Omega(n^2)$. However,

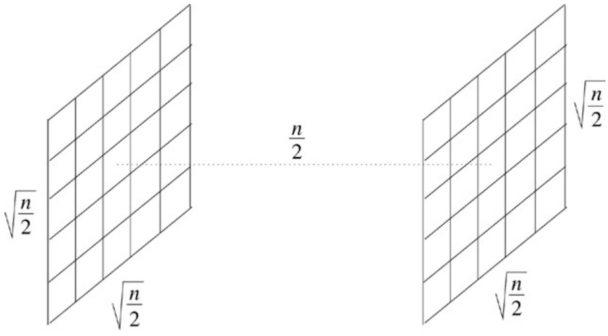

Fig. 2. An n-point separated set in $\mathbf{R}^3$ which determines $\frac{1}{4}n^2$ nearly equal distances and has diameter $O(n)$.

it is easy to find a set of n points in $\mathbf{R}^3$ with $\frac{n^2}{4}$ nearly equal distances, whose diameter is $O(n)$: Take two $\sqrt{\frac{n}{2}} \times \sqrt{\frac{n}{2}}$ integer grids in two parallel planes at distance $\frac{n}{2}$ from each other (see Fig. 2). Erdős conjectured that there exists no such example with diameter $o(n)$.

We prove the Erdős conjecture in the following more general form:

Theorem 1.1. *Let $d \geq 2$ and $\gamma > 0$ be fixed. Let P be a separated set of n points in $\mathbf{R}^d$ such that at least γn^2 pairs of points in P determine nearly equal distances. Then P has diameter at least $C(d, \gamma) n^{2/(d-1)}$ for some $C(d, \gamma) > 0$.*

The construction depicted on Fig. 2 can be easily generalized to higher dimensions, showing that the bound in Theorem 1.1 is tight. Our proof of Theorem 1.1 is based on Szemerédi's regularity lemma for dense graphs [15], and on a Ramsey-type result for dot products of vectors, derived in [1]. In Section 2, we reduce the problem to the "complete bipartite" case. That is, we show that it is sufficient to prove Theorem 1.1 for point sets P that can be obtained as the union of two sets Q and R such that all distances $\|\mathbf{u} - \mathbf{v}\|$ ($\mathbf{u} \in Q, \mathbf{v} \in R$) are nearly equal. At the end of Section 2, we outline the proof in this special case. The argument is divided into three steps, presented in full detail in Sections 3–5. For an alternative approach in three dimensions, see [18].

2. Reduction to the complete bipartite case

The following result shows that it is sufficient to establish Theorem 1.1 in the "complete bipartite case".

Theorem 2.1. *Let $\gamma > 0, t > 0$, and let P be a set of n points in $\mathbf{R}^d$ with at least γn^2 pairs $\{\mathbf{u}, \mathbf{v}\} \subset P$, such that $\|\mathbf{u} - \mathbf{v}\| \in [t, t+1]$.*

Then there exist two subsets $Q, R \subset P$ such that $|Q| = |R| \geq cn$ and $\|\mathbf{u} - \mathbf{v}\| \in [t, t+1]$ for all $\mathbf{u} \in Q, \mathbf{v} \in R$. (Here $c := c(d, \gamma)$ is a positive constant depending only on d and γ.)

Proof. Let $G = (V(G), E(G))$ be the graph on the vertex set $V(G) := P$ in which two vertices $\mathbf{u}, \mathbf{v} \in V(G)$ are connected by an edge if and only if $\|\mathbf{u} - \mathbf{v}\| \in [t, t+1]$. By the assumptions, we have $e(G) = |E(G)| \geq \gamma n^2$.

For any subsets $X, Y \subseteq V(G)$, let $e(X, Y)$ denote the number of edges of G with one endpoint in X and the other in Y. For any $\mathbf{v} \in V(G)$, let $\deg(v)$ stand for the degree of v in G.

In order to use Szemerédi's regularity lemma in the convenient and efficient form proposed by Komlós [15] (see also [13]), we have to introduce the notion of *super-regularity*.

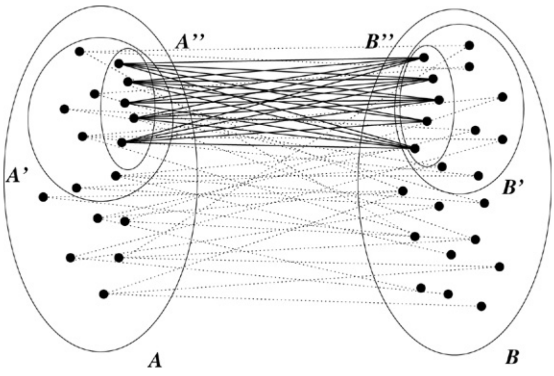

Fig. 3. Finding a complete bipartite subgraph in G.

Definition 2.2. Let $\varepsilon > 0$ and $\delta > 0$. Given a graph $G = (V, E)$ and two disjoint subsets $A, B \subseteq V$, we say that the pair $\{A, B\}$ is (ε, δ)-super-regular if the following two conditions are satisfied:

(i) $e(X, Y) > \delta|X| \cdot |Y|$ for every $X \subseteq A$, $Y \subseteq B$ with $|X| \geq \varepsilon|A|$, $|Y| \geq \varepsilon|B|$;
(ii) $\deg(a) \geq \delta|B|$ for all $a \in A$, and $\deg(b) \geq \delta|A|$ for all $b \in B$.

Lemma 2.3 (*Komlós*). *There exists a constant* ε_0 *such that if* $\varepsilon \leq \varepsilon_0$, $t = (3/\varepsilon)\log(1/\varepsilon)$, *and* G *is a graph with* n *vertices and* γn^2 *edges, then* G *contains an* (ε, δ)*-super-regular pair* (A, B) *with* $|A| = |B| \geq (2\gamma)^t \lfloor \frac{n}{2} \rfloor$ *and* $\delta \geq \gamma$.

Consider the graph G and set $\varepsilon = \min\{\frac{1}{4^{d+3}}, \varepsilon_0\}$. Using Lemma 2.3, we obtain an (ε, δ)-super-regular pair (A, B) with $|A| = |B| \geq (2\gamma)^t \lfloor \frac{n}{2} \rfloor$, $\delta \geq \gamma$, and $t = (3/\varepsilon)\log(1/\varepsilon)$. Define two maps $\omega_1, \omega_2 : A \cup B \mapsto \mathbf{R}^{d+2}$ as follows. For all $\mathbf{u} = (u_1, u_2, \ldots, u_d) \in A$, $\mathbf{v} = (v_1, v_2, \ldots, v_d) \in B$, let

$$\begin{aligned}
\omega_1(\mathbf{u}) &= (u_1, u_2, \ldots, u_d, \|\mathbf{u}\|^2 - t^2, 1),\\
\omega_2(\mathbf{u}) &= (u_1, u_2, \ldots, u_d, \|\mathbf{u}\|^2 - (t+1)^2, 1),\\
\omega_1(\mathbf{v}) &= (-2v_1, -2v_2, \ldots, -2v_d, 1, \|\mathbf{v}\|^2),\\
\omega_2(\mathbf{v}) &= (2v_1, 2v_2, \ldots, 2v_d, -1, -\|\mathbf{v}\|^2).
\end{aligned}$$

Then, for all $\mathbf{u} \in A$, $\mathbf{v} \in B$, the edge $\{\mathbf{u}, \mathbf{v}\}$ is in $E(G)$, that is, $\|\mathbf{u} - \mathbf{v}\| \in [t, t+1]$ if and only if $\omega_1(\mathbf{u}) \cdot \omega_1(\mathbf{v}) \geq 0$ and $\omega_2(\mathbf{u}) \cdot \omega_2(\mathbf{v}) \geq 0$ (see Fig. 3).

We need the following lemma of Alon et al. [1] that can be established using a consequence of the Borsuk–Ulam theorem discovered by Yao and Yao [23].

Lemma 2.4 (*Alon et al.*). *Let* U *and* V *be finite sets of vectors in* $\mathbf{R}^k$. *Then there exist subsets* $U' \subset U$, $V' \subset V$ *with* $|U'| \geq \frac{1}{2^{k+1}}|U|$, $|V'| \geq \frac{1}{2^{k+1}}|V|$ *such that either* $\mathbf{u} \cdot \mathbf{v} \geq 0$ *holds for all* $\mathbf{u} \in U'$, $\mathbf{v} \in V'$, *or* $\mathbf{u} \cdot \mathbf{v} < 0$ *holds for all* $\mathbf{u} \in U'$, $\mathbf{v} \in V'$.

Applying this lemma with $k = d + 2$ to the sets $U := \omega_1(A)$, $V := \omega_1(B)$, we obtain two subsets $A' \subset A$, $B' \subset B$ such that $|A'| \geq \frac{1}{2^{d+3}}|A|$, $|B'| \geq \frac{1}{2^{d+3}}|B|$, and either $\omega_1(\mathbf{u}) \cdot \omega_1(\mathbf{v}) \geq 0$ holds for all $\mathbf{u} \in A'$, $\mathbf{v} \in B'$, or $\omega_1(\mathbf{u}) \cdot \omega_1(\mathbf{v}) < 0$ holds for all $\mathbf{u} \in A'$, $\mathbf{v} \in B'$. Observe that this corresponds to $\|\mathbf{u} - \mathbf{v}\| \geq t$ or $\|\mathbf{u} - \mathbf{v}\| < t$.

Applying the same once again to $U' = \omega_2(A')$ and $V' = \omega_2(B')$, we obtain subsets $A'' \subset A'$, $B'' \subset B'$ of size $|A''| \geq \frac{1}{2^{d+3}}|A'|$, $|B''| \geq \frac{1}{2^{d+3}}|B'|$ such that either $\omega_2(\mathbf{u}) \cdot \omega_2(\mathbf{v}) \geq 0$ holds for all $\mathbf{u} \in A''$, $\mathbf{v} \in B''$, or $\omega_2(\mathbf{u}) \cdot \omega_2(\mathbf{v}) < 0$ holds for all $\mathbf{u} \in A''$, $\mathbf{v} \in B''$. Consequently the pairwise distances $\|\mathbf{u} - \mathbf{v}\|$ for $\mathbf{u} \in A''$, $\mathbf{v} \in B''$ are either all in $[0, t)$, all in $[t, t+1]$ or all in $(t+1, \infty)$.

We claim that the pairwise distances between A'' and B'' must be all in $[t, t+1]$. If this were not the case, they would be all outside of $[t, t+1]$ and we would have $e(A'', B'') = 0$. However, by the (ε, δ)-super-regularity of the pair (A, B), we obtain $e(A'', B'') > \delta|A''| \cdot |B''| > 0$, since $\varepsilon = \min\{\frac{1}{4^{d+3}}, \varepsilon_0\}$ and $|A''| \geq \frac{1}{2^{d+3}}|A'| \geq \frac{1}{4^{d+3}}|A| \geq \varepsilon|A|$, $|B''| \geq \frac{1}{2^{d+3}}|B'| \geq \frac{1}{4^{d+3}}|B| \geq \varepsilon|B|$.

Thus, we conclude that $\|\mathbf{u} - \mathbf{v}\| \in [t, t+1]$ for all $\mathbf{u} \in A''$, $\mathbf{v} \in B''$. Furthermore, both A'' and B'' are of size at least $\varepsilon|A| = \varepsilon|B| \geq \varepsilon(2\gamma)^t \lfloor \frac{n}{2} \rfloor$, where $\varepsilon = \min\{\frac{1}{4^{d+3}}, \varepsilon_0\}$ and $t = (3/\varepsilon)\log(1/\varepsilon) = O(d4^d)$. Consequently, the sets $Q := A''$ and $R := B''$ meet the requirements of Theorem 2.1. The constant factor $c(d, \gamma)$ is roughly $\gamma^{O(d4^d)}$. □

It remains to establish Theorem 1.1 for separated point sets that can be partitioned into two parts Q and R of size m such that all pairs belonging to $Q \times R$ determine nearly equal distances. The argument is divided into three steps.

1. In the first step, described in Section 3, we select a set $T \subset R$ of at most $2d$ points, spanning a "fat crosspolytope" with near-orthogonal axes. The "fatness" of T is measured by a certain determinant $D(T)$ (which corresponds to the volume of the crosspolytope assuming it is convex). We show that there is a set $T \subset R$ with $D(T) = \Omega(|R|) = \Omega(m)$. The existence of T relies heavily on the assumption that R is a separated point set.
2. In the second step (Section 4), we bound the volume of the locus of points whose distance from each vertex of T belongs to the interval $[t, t+1]$. Note that this region can be obtained as the intersection of $|T|$ spherical annuli centered at the vertices of T. We show that this intersection has volume $O(t^{d-1}/D(T))$.
3. In Section 5, we complete the proof of Theorem 1.1 by observing that Q is contained in the region discussed in Section 4, whose volume is $O(t^{d-1}/D(T)) = O(t^{d-1}/m)$. Since Q is a separated set of size m, the volume of this region must be $\Omega(m)$. This implies $t = \Omega(m^{2/(d-1)})$.

3. Finding a fat crosspolytope

First, we consider only one part of the bipartite subgraph, R, and we find a small subset $T \subset R$ which spans a sufficiently "fat" crosspolytope. In this section, we are not using the condition of nearly equal distances, only the fact that R is a separated set. The following is our measure of "fatness".

Definition 3.1. Given a set $T = \{\mathbf{p_1}, \mathbf{q_1}, \ldots, \mathbf{p_r}, \mathbf{q_r}\}$, consisting of r pairs of distinct points, let $D(T)$ denote the determinant

$$[\mathbf{q_1} - \mathbf{p_1}, \ldots, \mathbf{q_r} - \mathbf{p_r}, \mathbf{e_{r+1}}, \ldots, \mathbf{e_d}]$$

where $\mathbf{e_{r+1}}, \ldots, \mathbf{e_d}$ are mutually orthogonal and also orthogonal to $\mathbf{q_1} - \mathbf{p_1}, \ldots, \mathbf{q_r} - \mathbf{p_r}$. For $T = \emptyset$, we set $D(T) = 1$.

Note that $\frac{1}{r!}D(T)$ is the r-dimensional volume of the convex hull of T, provided that the points $\{\mathbf{p_1}, \mathbf{q_1}, \ldots, \mathbf{p_r}, \mathbf{q_r}\}$ are in a convex position. However, in the sequel this fact will not be used.

Now we can formalize the first step of the proof outlined at the end of the last section. First, we need an elementary lemma bounding the size of a separated set in a given volume.

Lemma 3.2. *Let $X \subset \mathbf{R}^d$ be a separated set of points, and let $B_{1/2}(x)$ denote a ball of radius $1/2$ centered at x. If $B_{1/2}(x) \subset Z$ for every $x \in X$, then*

$$|X| < d^{d/2}\, Vol(Z).$$

Proof. The balls $B_{1/2}(x)$ are pairwise disjoint for all $x \in X$. Their union is contained in Z, therefore

$$Vol(Z) \geq \sum_{x \in X} Vol(B_{1/2}(x)) = \frac{\pi^{d/2}}{2^d \Gamma(1 + d/2)} |X| > \frac{1}{d^{d/2}} |X|,$$

where the last inequality follows from standard estimates for the Gamma function [5]. □

The main result of this section is the following.

Lemma 3.3. *Let $R \subset \mathbf{R}^d$ be a separated set of m points, of diameter Δ. Let $\delta > 1$ and $\alpha = 1/(k\sqrt{d})$ for some $k \in \mathbf{Z}_+$. Then there is an orthonormal basis $\{\mathbf{e_1}, \ldots, \mathbf{e_d}\}$ and points $\{\mathbf{p_1}, \mathbf{q_1}, \ldots, \mathbf{p_r}, \mathbf{q_r}\} = T \subseteq R$ (for some $0 \leq r \leq d$; T possibly empty) such that*

1. *For all $k \leq r$ we have $\mathbf{q_k} - \mathbf{p_k} = h_k \mathbf{e_k} + \sum_{j=1}^{k-1} \beta_{jk} \mathbf{e_j}$, where $h_k \geq \delta$, $|\beta_{jk}| \leq 1$.*
2. *$D(T) \geq \left(\frac{\alpha}{d(\delta+3)}\right)^d m$.*
3. *The diameter of T is at most $\alpha\Delta$.*

Proof. First, we "reduce" the diameter of R, which must be contained in a hypercube $\mathcal{H}$ of side length Δ. We partition $\mathcal{H}$ into subcubes of diameter $\alpha\Delta$. This can be accomplished, for example, by choosing $a = \sqrt{d}/\alpha = kd$ and subdividing $\mathcal{H}$ uniformly into a^d subcubes of side length $\Delta/a = \alpha\Delta/\sqrt{d}$. (Note that we are using very rough estimates; we make no attempt to optimize multiplicative factors depending only on d.) By the pigeonhole principle, there is a subset $R_1 \subseteq R$ such that

1. $|R_1| = n_1 \geq m/a^d$,
2. $\mathrm{diam}(R_1) \leq \alpha\Delta$.

Let $\{\mathbf{p_1}, \mathbf{q_1}\}$ be a pair of points at maximal distance in R_1 and let $h_1 = \|\mathbf{q_1} - \mathbf{p_1}\|$. If $h_1 < \delta$ then we stop, set $T = \emptyset$, and choose an arbitrary orthonormal basis $\{\mathbf{e_1}, \ldots, \mathbf{e_d}\}$. In this case, R_1 is contained in a hypercube of side length δ, which means (by Lemma 3.2) that $n_1 < d^{d/2}(\delta+1)^d$ and $m \leq a^d n_1 < (d(\delta+1)/\alpha)^d$, so the statement of the lemma is true.

Otherwise, let $\mathbf{e_1} = (\mathbf{q_1} - \mathbf{p_1})/h_1$. Note that for any point $\mathbf{x} \in R_1$, we have $\mathbf{x} \cdot \mathbf{e_1} \in [\mathbf{p_1} \cdot \mathbf{e_1}, \mathbf{q_1} \cdot \mathbf{e_1}]$, which is an interval of size h_1. We assume, for simplicity, that h_1 is an integer, and subdivide the interval $[\mathbf{p_1} \cdot \mathbf{e_1}, \mathbf{q_1} \cdot \mathbf{e_1}]$ into h_1 unit intervals. By the pigeonhole principle, there is a subset $R_2 \subseteq R_1$ such that

1. $n_2 = |R_2| \geq n_1/h_1$, and
2. there exists b_1 such that for all $\mathbf{x} \in R_2$ we have $\mathbf{x} \cdot \mathbf{e_1} \in [b_1, b_1 + 1]$.

We continue this procedure, restricting our attention to the subspace orthogonal to the previously constructed pairs of points. For $k > 1$, assume that we have constructed vectors $\mathbf{e_1}, \ldots, \mathbf{e_{k-1}}$ and subsets $R_1, \ldots, R_k$. Denote by $\mathcal{S}_{k-1}$ the subspace generated by $\{\mathbf{e_1}, \ldots, \mathbf{e_{k-1}}\}$

and by $\mathcal{S}_{k-1}^{\perp}$ its orthogonal complement. Assume that the diameter of R_k projected on $\mathcal{S}_{k-1}^{\perp}$ is $h_k \geq \delta$. Namely, there is a unit vector $\mathbf{e_k}$ orthogonal to $\mathbf{e_1}, \ldots, \mathbf{e_{k-1}}$, and there are extreme points $\mathbf{p_k}, \mathbf{q_k} \in R_k$ such that

$$\mathbf{q_k} - \mathbf{p_k} = h_k \mathbf{e_k} + \sum_{j=1}^{k-1} \beta_{jk} \mathbf{e_j} \tag{1}$$

for some $|\beta_{jk}| \leq 1$. In addition, for every $\mathbf{x} \in R_k$, we have $\mathbf{x} \cdot \mathbf{e_k} \in [\mathbf{p_k} \cdot \mathbf{e_k}, \mathbf{q_k} \cdot \mathbf{e_k}]$. Again, there must be a subset $R_{k+1} \subseteq R_k$ such that

1. $n_{k+1} = |R_{k+1}| \geq n_k / h_k$, and
2. there exists b_k such that for all $\mathbf{x} \in R_{k+1}$ we have $\mathbf{x} \cdot \mathbf{e_k} \in [b_k, b_k + 1]$.

Iterate this procedure as long as $h_k \geq \delta$. Let r be the minimum index such that $h_{r+1} < \delta$. If $h_1, h_2, \ldots, h_d \geq \delta$, we set $h_{d+1} = 0$ and $r = d$. If $r < d$, choose $d - r$ additional unit vectors so that we have an orthonormal basis $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_d}\}$.

We set $T = \{\mathbf{p_1}, \mathbf{q_1}, \ldots, \mathbf{p_r}, \mathbf{q_r}\}$. It remains to estimate the determinant $D(T)$. Note that due to (1), we have

$$D(T) = [\mathbf{q_1} - \mathbf{p_1}, \ldots, \mathbf{q_r} - \mathbf{p_r}, \mathbf{e_{r+1}}, \ldots, \mathbf{e_d}] = h_1 h_2 \ldots h_r. \tag{2}$$

Since $h_{r+1} < \delta$, R_{r+1} must be contained in a hypercube of side length δ and volume $(\delta + 1)^d$. By Lemma 3.2, we obtain $n_{r+1} = |R_{r+1}| < d^{d/2}(\delta + 1)^d$. On the other hand, we have

$$n_{r+1} \geq \frac{n_r}{h_r} \geq \frac{n_{r-1}}{h_{r-1} h_r} \geq \cdots \geq \frac{n_1}{h_1 h_2 \ldots h_r} \geq \frac{m}{a^d h_1 \ldots h_r}.$$

We assumed that each h_k is an integer; in general, we should consider $\lceil h_k \rceil$ and partition each interval $[b_k, b_k + 1]$ into $\lceil h_k \rceil \leq h_k(1 + 1/\delta)$ subintervals. This does not make any significant difference; in general, we have $n_{r+1} \geq m/(a^d (1 + 1/\delta)^d h_1 \ldots h_r)$. Finally using (2), we obtain

$$D(T) = h_1 h_2 \ldots h_r \geq \frac{m}{a^d d^{d/2} (1 + 1/\delta)^d (\delta + 1)^d} \geq \left(\frac{\alpha}{d(\delta + 3)} \right)^d m. \quad \square \tag{3}$$

4. Intersecting the annuli

In the second step of the proof outlined at the end of Section 2, we use the crosspolytope T constructed in Section 3 to restrict the region of possible locations for the points in Q. These points must be at distance between t and $t + 1$ from each vertex of T; this defines an annulus containing Q, for each vertex of T. In fact, we consider an interval of distances $[t - \frac{1}{2}, t + \frac{3}{2}]$, in order to contain not only Q but also a ball of radius $1/2$ around each point in Q. First, we analyze the intersection of two such annuli.

Lemma 4.1. *Let $\|\mathbf{p} - \mathbf{q}\| = h$. Define an annulus*

$$An(\mathbf{y}) = \left\{ \mathbf{x} \in \mathbf{R}^d : \|\mathbf{x} - \mathbf{y}\| \in \left[t - \frac{1}{2}, t + \frac{3}{2} \right] \right\}.$$

Then the intersection of $An(\mathbf{p}) \cap An(\mathbf{q})$ is contained in a "slab" of thickness $(4t + 2)/h$ defined by

$$L(\mathbf{p}, \mathbf{q}) = \left\{ \mathbf{x} \in \mathbf{R}^d : \left(\mathbf{x} - \frac{\mathbf{p} + \mathbf{q}}{2} \right) \cdot (\mathbf{q} - \mathbf{p}) \in [-2t - 1, 2t + 1] \right\}.$$

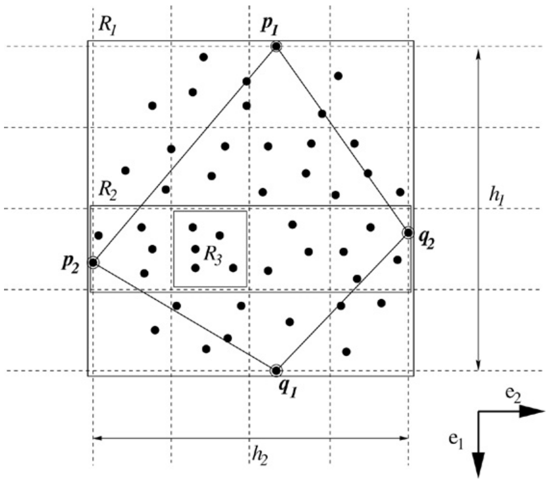

Fig. 4. The first two steps of constructing the fat polytope (projection onto $\mathcal{S}_2$). R_3 will be the set of points considered in the next stage.

Proof. Assume $\|\mathbf{x}-\mathbf{p}\|, \|\mathbf{x}-\mathbf{q}\| \in [t-\frac{1}{2}, t+\frac{3}{2}]$. We have

$$\left(\mathbf{x}-\frac{\mathbf{p}+\mathbf{q}}{2}\right)\cdot(\mathbf{q}-\mathbf{p}) = \frac{1}{2}\|\mathbf{x}-\mathbf{p}\|^2 - \frac{1}{2}\|\mathbf{x}-\mathbf{q}\|^2 \le \frac{1}{2}\left(t+\frac{3}{2}\right)^2 - \frac{1}{2}\left(t-\frac{1}{2}\right)^2 = 2t+1.$$

Similarly, $\left(\mathbf{x}-\frac{\mathbf{p}+\mathbf{q}}{2}\right)\cdot(\mathbf{q}-\mathbf{p}) \ge -2t-1$. □

With the help of this lemma, we are now able to bound the intersection of the annuli centered at each point of T (see Fig. 4).

Lemma 4.2. *Suppose that* $T = \{\mathbf{p_1}, \mathbf{q_1}, \ldots, \mathbf{p_r}, \mathbf{q_r}\}$ *is a set of points as guaranteed by Lemma 3.3, for* $\Delta = 2(t+1)$, $\alpha = 1/(16\sqrt{d})$, $\delta = \max\{2d, 16\sqrt{d}\}$, *and* $t \ge 3$. *For any* $\mathbf{y} \in T$, *define the annulus* $An(\mathbf{y})$ *centered at* $\mathbf{y}$ *as in Lemma 4.1. Then we have*

$$Vol\left(\bigcap_{i=1}^{r}(An(\mathbf{p_i}) \cap An(\mathbf{q_i}))\right) \le \frac{100(4t+2)^{d-1}}{D(T)}.$$

Proof. Instead of directly analyzing the intersection of the above annuli, we apply Lemma 4.1. Consider the region

$$\mathcal{R} = L(\mathbf{p_1}, \mathbf{q_1}) \cap L(\mathbf{p_2}, \mathbf{q_2}) \cap \cdots \cap L(\mathbf{p_r}, \mathbf{q_r}).$$

By Lemma 4.1,

$$\mathcal{R} = \left\{\mathbf{x} \in \mathbf{R}^d : \left(\mathbf{x}-\frac{\mathbf{p_i}+\mathbf{q_i}}{2}\right)\cdot(\mathbf{q_i}-\mathbf{p_i}) \in [-2t-1, 2t+1] \text{ for all } i = 1, \ldots, r\right\}.$$

Since the vectors $\mathbf{q_1}-\mathbf{p_1}, \ldots, \mathbf{q_r}-\mathbf{p_r}$ are in the subspace $\mathcal{S}_r$ generated by $\{\mathbf{e_1}, \ldots, \mathbf{e_r}\}$, we can regard $\mathcal{R}$ as a prism with an r-dimensional base $\mathcal{R}_r \subset \mathcal{S}_r$, extending indefinitely in the orthogonal subspace $\mathcal{S}_r^{\perp}$ generated by $\{\mathbf{e_{r+1}}, \ldots, \mathbf{e_d}\}$.

Next, we calculate the r-dimensional volume of the base $\mathcal{R}_r$. This is the same as the volume of $\mathcal{R}$ confined to a unit hypercube in $\mathcal{S}_r^{\perp}$:

$$\bar{\mathcal{R}} = \left\{ \mathbf{x} \in \mathbf{R}^d : \left(\mathbf{x} - \frac{\mathbf{p_i} + \mathbf{q_i}}{2} \right) \cdot (\mathbf{q_i} - \mathbf{p_i}) \in [-2t-1, 2t+1] \text{ for all } i = 1, 2, \ldots, r \right.$$
$$\left. \text{and } \mathbf{x} \cdot \mathbf{e_i} \in [0, 1] \text{ for all } i = r+1, \ldots, d \right\}.$$

$\bar{\mathcal{R}}$ is a set that maps to a hypercube of volume $(4t + 2)^r$ via an affine transformation whose Jacobian is $D(T)$. Therefore

$$Vol_r(\mathcal{R}_r) = Vol(\bar{\mathcal{R}}) = (4t + 2)^r / D(T). \tag{4}$$

Finally, we intersect $\mathcal{R}$ once again with an annulus centered at a point of T, for example, with $An(\mathbf{p_1})$. In order to bound the volume of $\mathcal{R} \cap An(\mathbf{p_1})$, we need to argue that $\mathcal{R}$ is located relatively close to $\mathbf{p_1}$. For any point $\mathbf{x} \in \mathcal{R}$ and for any $k = 1, 2, \ldots, r$, we have

$$\begin{aligned} &\left|(\mathbf{x} - \mathbf{p_1}) \cdot (\mathbf{q_k} - \mathbf{p_k})\right| \\ &\leq \left| \left(\mathbf{x} - \frac{\mathbf{p_k} + \mathbf{q_k}}{2} \right) \cdot (\mathbf{q_k} - \mathbf{p_k}) \right| + \left| \left(\frac{\mathbf{p_k} + \mathbf{q_k}}{2} - \mathbf{p_1} \right) \cdot (\mathbf{q_k} - \mathbf{p_k}) \right| \\ &\leq (2t + 1) + \left\| \frac{\mathbf{p_k} + \mathbf{q_k}}{2} - \mathbf{p_1} \right\| h_k \leq 2(t + 1)(1 + \alpha h_k), \end{aligned} \tag{5}$$

using the definition of $\mathcal{R}$ and the fact that the diameter of T is bounded by $\alpha\Delta = 2\alpha(t + 1)$. We claim that for any $k \leq r$

$$|(\mathbf{x} - \mathbf{p_1}) \cdot \mathbf{e_k}| \leq 4(t + 1) \left(\frac{1}{h_k} + \alpha \right) \tag{6}$$

holds. Consider the index k maximizing $\left|(\mathbf{x} - \mathbf{p_1}) \cdot \mathbf{e_k}\right|$, and assume on the contrary that $\left|(\mathbf{x} - \mathbf{p_1}) \cdot \mathbf{e_k}\right| > 4(t + 1)(1/h_k + \alpha)$. Recall (1). In terms of the basis $\{\mathbf{e_1}, \ldots, \mathbf{e_d}\}$, we can write $\mathbf{q_k} - \mathbf{p_k} = h_k \mathbf{e_k} + \sum_{j<k} \beta_{jk} \mathbf{e_j}$, where $h_k \geq \delta$ and $|\beta_{jk}| \leq 1$. We obtain

$$\begin{aligned} |(\mathbf{x} - \mathbf{p_1}) \cdot (\mathbf{q_k} - \mathbf{p_k})| &= \left| (\mathbf{x} - \mathbf{p_1}) \cdot \left(h_k \mathbf{e_k} + \sum_{j=1}^{k-1} \beta_{jk} \mathbf{e_j} \right) \right| \\ &\geq |h_k (\mathbf{x} - \mathbf{p_1}) \cdot \mathbf{e_k}| - \sum_{j=1}^{k-1} |(\mathbf{x} - \mathbf{p_1}) \cdot \mathbf{e_j}| \\ &\geq (h_k - (k - 1)) |(\mathbf{x} - \mathbf{p_1}) \cdot \mathbf{e_k}|, \end{aligned}$$

using the maximality of $|(\mathbf{x} - \mathbf{p_1}) \cdot \mathbf{e}_k|$. Finally, taking into account that $h_k \geq \delta \geq 2(k - 1)$, we have

$$|(\mathbf{x} - \mathbf{p_1}) \cdot (\mathbf{q_k} - \mathbf{p_k})| > \frac{h_k}{2} |(\mathbf{x} - \mathbf{p_1}) \cdot \mathbf{e_k}| > 2(t + 1)(1 + \alpha h_k)$$

which contradicts (5). This proves (6). We assume $h_k \geq \delta \geq 16\sqrt{d}$ and we choose $\alpha = 1/(16\sqrt{d})$, which implies that

$$|(\mathbf{x} - \mathbf{p_1}) \cdot \mathbf{e_k}| \leq \frac{t + 1}{2\sqrt{d}} \quad \text{for all } \mathbf{x} \in \mathcal{R} \text{ and for all } k \leq r. \tag{7}$$

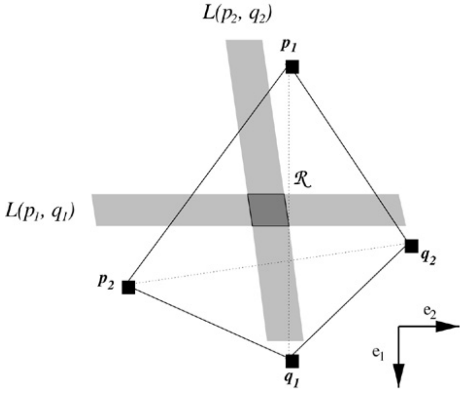

Fig. 5. The intersection prism $\mathcal{R} = L(\mathbf{p_1}, \mathbf{q_1}) \cap L(\mathbf{p_2}, \mathbf{q_2})$ (projection onto $\mathcal{S}_2$, $r = 2$). The prism extends indefinitely in the dimensions orthogonal to $\mathcal{S}_2$.

Without loss of generality, assume that the base $\mathcal{R}_r$ is translated along the prism $\mathcal{R}$ so that its r-dimensional affine hull contains $\mathbf{p_1}$. Then every point $\mathbf{x} \in \mathcal{R}_r$ satisfies $(\mathbf{x} - \mathbf{p_1}) = \sum_{j=1}^{r}((\mathbf{x} - \mathbf{p_1}) \cdot \mathbf{e_j})\mathbf{e_j}$, and

$$\|\mathbf{x} - \mathbf{p_1}\|^2 = \sum_{j=1}^{r}((\mathbf{x} - \mathbf{p_1}) \cdot \mathbf{e_j})^2 \leq \frac{r(t+1)^2}{4d} \leq \frac{(t+1)^2}{4}.$$

Thus, every point of $\mathcal{R}_r$ is a distance at most $(t+1)/2$ from $\mathbf{p_1}$ (see Fig. 5).

Now we are ready to estimate the volume of $\mathcal{R} \cap An(\mathbf{p_1})$. Write $\mathcal{R} = \bigcup_{\mathbf{x} \in \mathcal{R}_r}(\mathbf{x} + \mathcal{S}_r^{\perp})$, where $(\mathbf{x} + \mathcal{S}_r^{\perp})$ denotes an affine subspace through $\mathbf{x}$, orthogonal to $\mathcal{S}_r$. Notice that $An(\mathbf{p_1}) \cap (\mathbf{x} + \mathcal{S}_r^{\perp})$ is a $(d-r)$-dimensional annulus, or the region between $(d-r-1)$-dimensional spheres of radii $r_1 = \sqrt{(t-1/2)^2 - \rho^2}$ and $r_2 = \sqrt{(t+3/2)^2 - \rho^2}$, where $\rho = \|\mathbf{x} - \mathbf{p_1}\| \leq (t+1)/2$. Let $S^{(d-r-1)}$ denote a $(d-r-1)$-dimensional unit sphere. We get

$$\begin{aligned} Vol_{d-r}(An(\mathbf{p_1}) \cap (\mathbf{x} + \mathcal{S}_r^{\perp})) &= \int_{r_1}^{r_2} z^{d-r-1} Vol_{d-r-1}(S^{(d-r-1)})\,\mathrm{d}z \\ &\leq (r_2 - r_1) r_2^{d-r-1}\, Vol_{d-r-1}(S^{(d-r-1)}). \end{aligned}$$

We have $\rho \leq (t+1)/2$ and

$$r_2 - r_1 \leq \sqrt{(t+3/2)^2 - (t+1)^2/4} - \sqrt{(t-1/2)^2 - (t+1)^2/4},$$

which can be verified to be bounded from above by 3 for $t \geq 3$. The volume of $S^{(d-r-1)}$ is bounded by 33 in any dimension [5]. We obtain the volume of $\mathcal{R} \cap An(\mathbf{p_1})$ by integrating over all $\mathbf{x} \in \mathcal{R}_r$:

$$\begin{aligned} Vol(\mathcal{R} \cap An(\mathbf{p_1})) &= \int_{\mathcal{R}_r} Vol_{d-r}(An(\mathbf{p_1}) \cap (\mathbf{x} + \mathcal{S}_r^{\perp}))\,\mathrm{d}\mathbf{x} \\ &< 100 \int_{\mathcal{R}_r} r_2^{d-r-1}\,\mathrm{d}\mathbf{x} \leq 100(t+3/2)^{d-r-1}\frac{(4t+2)^r}{D(T)} \end{aligned}$$

$$\leq \frac{100(4t+2)^{d-1}}{D(T)},$$

using the volume of $\mathcal{R}_r$ from (4). □

5. Proof of Theorem 1.1 and concluding remarks

Now we can complete the proof of Theorem 1.1 in the complete bipartite case.

Theorem 5.1. *Let $Q \cup R \subset \mathbf{R}^d$ be a separated set of points such that $|Q| = |R| = m$ and all distances between $\mathbf{x} \in Q$ and $\mathbf{y} \in R$ are between t and $t+1$. Then there is a constant $C_d > 0$ such that*

$$t > (C_d - o(1))\, m^{2/(d-1)}.$$

Proof. We can assume that $t \geq 3$. (For $t < 3$, there is only a constant number of points in Q that can fit within distance $t+1$ from any $\mathbf{y} \in R$.) Note also that the diameter of R is at most $\Delta = 2(t+1)$, due to the condition of nearly equal distances between Q and R. Then the balls of radius $1/2$ centered at each point of Q are disjoint and, by Lemma 4.2, must be contained in a region of volume $V \leq 100(4t+2)^{d-1}/D(T)$. Lemma 3.3 with $\alpha = 1/(16\sqrt{d})$ implies that $D(T) \geq m/(16d^{3/2}(\delta+3))^d$. By Lemma 3.2, we obtain

$$m = |Q| < d^{d/2} \cdot 100(4t+2)^{d-1} \frac{(16d^{3/2}(\delta+3))^d}{m},$$

$$(4t+2)^{d-1} > \frac{m^2}{100(16d^2(\delta+3))^d},$$

where $\delta = \max\{2d, 16\sqrt{d}\}$. Asymptotically (for d fixed and $m \to \infty$), we have

$$t > (C_d - o(1))\, m^{2/(d-1)}.$$

For large d, the multiplicative constant C_d is roughly $1/(128d^3)$. □

Together with Theorem 2.1, this proves the main result, Theorem 1.1.

Since Theorem 2.1 provides only $m \geq c(d,\gamma)n$ where $c(d,\gamma) \geq \gamma^{O(d4^d)}$, the loss of factor $O(d^3)$ in Theorem 5.1 is insignificant. The constant factor that we obtain for Theorem 1.1 is $C(\gamma, d) \geq \gamma^{O(4^d)}$, i.e. doubly exponentially small in d. We did not try to optimize this constant.

We could have used Szemerédi's original regularity lemma in place of Lemma 2.3. However, this would have given a much smaller regular pair (A, B) of density roughly γ: its size would have been only about $n/\text{tower}(1/\gamma)$ (a tower function of $1/\gamma$). It was shown in [1] that the $\frac{1}{2^{k+1}}$-factor in Lemma 2.4 cannot be substantially improved.

In [1], Lemma 2.4 was used to establish the existence of a positive constant β such that every family $\mathcal{F}$ of n semi-algebraic sets in $\mathbf{R}^d$ of constant description complexity has two subfamilies $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{F}$, each containing at least βn members, with the property that every member of $\mathcal{F}_1$ intersects all members of $\mathcal{F}_2$ or no member of $\mathcal{F}_1$ intersects any member of $\mathcal{F}_2$. For other geometric consequences of Lemma 2.4, consult [1]. We believe that Lemma 2.4, in combination with other ideas, such as the regularity lemma, may be a useful tool for various other problems in discrete geometry and Ramsey theory.

Finally, we mention a related open problem of Erdős. Let P be a set of n points in $\mathbf{R}^d$. We call P *admissible* if the unit distance is the minimum distance determined by P and any two different distances determined by P differ by *at least* 1. Erdős asked for the minimum diameter of an n-element admissible set in $\mathbf{R}^d$. For large n, it is known that the minimum is at least $c_d \cdot n^{1/(d-1)}$. On the other hand, there exist admissible sets with diameter at most $C_d \cdot n^{2/(d-1)}$ [4].

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