A note on coloring line arrangements

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Abstract

We show that the lines of every arrangement of $n$ lines in the plane can be colored with $O(\sqrt{n}/\log n)$ colors such that no face of the arrangement is monochromatic. This improves a bound of Bose et al. by a $\Theta(\sqrt{\log n})$ factor. Any further improvement on this bound would also improve the best known lower bound on the following problem of Erdős: estimate the maximum number of points in general position within a set of $n$ points containing no four collinear points.

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1 Introduction

Given a simple arrangement $\mathcal{A}$ of a set $L$ of lines in $\mathbb{R}^2$ (no parallel lines and no three lines going through the same point), decomposing the plane into the set $C$ of cells (i.e. maximal connected components of $\mathbb{R}^2 \setminus L$), Bose et al. [1] defined a hypergraph $H_{\text{line-cell}} = (L, C)$ with the vertex set $L$ (the set of lines of $\mathcal{A}$), and each hyperedge $c \in C$ being defined by the set of lines forming the boundary of a cell of $\mathcal{A}$. They initiated the study of the chromatic number of $H_{\text{line-cell}}$, and proved that for $|L| = n$, $\chi(H_{\text{line-cell}}) = O(\sqrt{n})$ and $\chi(H_{\text{line-cell}}) = \Omega(\log n \log \log n)$. In other words, they proved that the lines of every simple arrangement of $n$ lines can be colored with $O(\sqrt{n})$ colors so that there is no monochromatic face; furthermore, they provided an intricate construction of a simple arrangement of $n$ lines that requires $\Omega(\log n \log \log n)$ colors.

In this short note, we improve their upper bound by a $\Theta(\sqrt{\log n})$ factor, and extend it to not necessarily simple arrangements.

**Theorem 1.** The lines of every arrangement of $n$ lines in the plane can be colored with $O(\sqrt{n/\log n})$ colors so that no face of the arrangement is monochromatic.

A set of points in the plane is in general position if it does not contain three collinear points. Let $\alpha(S)$ denote the maximum number of points in general position in a set $S$ of points in the plane, and let $\alpha_4(n)$ be the minimum of $\alpha(S)$ taken over all sets $S$ of $n$ points in the plane with no four point on a line. Erdős pointed out that $\alpha_4(n) \leq n/3$ and suggested the problem of determining or estimating $\alpha_4(n)$. Füredi [3] proved that $\Omega(\sqrt{n \log n}) \leq \alpha_4(n) \leq o(n)$.

We observe that any improvement of the bound in Theorem 1 would immediately imply a better lower bound for $\alpha_4(n)$. Indeed, suppose that $\chi(A) \leq k(n)$ for any arrangement of $n$ lines, and let $P$ be a set of $n$ points, no four on a line. Let $P^*$ be the dual arrangement of a slightly perturbed $P$ (according to the usual point-line duality, see, e.g., [2, §8.2]). Color $P^*$ with $k(n)$ colors such that no face is monochromatic, let $S^* \subseteq P^*$ be the largest color class, and let $S$ be its dual point set. Observe that the size of $S$ is at least $n/k(n)$ and it does not contain three collinear points, since the three lines that correspond to any three collinear points in $P$ bound a face of size three in $P^*$.

2 Proof of Theorem 1

Let $\mathcal{A}$ be an arrangement of a set $L$ of $n$ lines, decomposing the plane into the set $C$ of cells, and let $H_{\text{line-cell}}$ be the corresponding hypergraph (defined as in the previous section).

We show that $\chi(H_{\text{line-cell}}) = O\left(\sqrt{\frac{n}{\log n}}\right)$.

An independent set in $H_{\text{line-cell}}$ is a set $S \subseteq L$ such that for every $c \in C$, $c$ is not a subset of $S$ (in other words, no cell of $\mathcal{A}$ has its boundary formed only by lines in $S$). The proof is based on the following fact.
Theorem 2. There is an absolute constant \( c > 0 \) such that the size \( \alpha(H_{\text{line-cell}}) \) of the maximum independent set is at least \( c\sqrt{n \log n} \).

We color the lines in \( A \) so that no face is monochromatic by following the same method as in [1] (where they used the weaker version of Theorem 2 stating \( \alpha(H_{\text{line-cell}}) = \Omega(\sqrt{n}) \)). That is, we iteratively find a large independent set of lines (whose existence is guaranteed by Theorem 2), color them with the same (new) color, and remove them from \( A \).

Clearly, this algorithm produces a valid coloring. We verify, by induction on \( n \), that at most \( \frac{2}{c}\sqrt{n \log n} \) colors are used in this coloring. We assume the bound is valid for all \( n \leq 256 \) (by taking sufficiently small \( c > 0 \)). For \( n > 256 \), we have \( \log 4 < \frac{1}{4} \log n \). Let \( i \) be the smallest integer such that after \( i \) iterations the number of remaining lines is at most \( n/4 \). Since in each of these iterations at least \( c\sqrt{\frac{n}{4}} \log n \geq c\sqrt{n/\log n} \) vertices (lines) are removed, \( i \leq \frac{n^4}{c^2 \log n} \leq \frac{1}{2c} \sqrt{n \log n} \). Therefore, by the induction hypothesis the number of colors that the algorithm uses is at most

\[
i + \frac{2}{c} \sqrt{\frac{n}{4} \log \frac{n}{4}} \leq \frac{1}{2c} \sqrt{n \log n} + \frac{1}{c} \sqrt{n \log n - \frac{n}{4} \log n} < \frac{1}{\sqrt{2c}} \sqrt{n \log n} + \frac{\sqrt{4/3}}{c} \sqrt{n \log n} \leq \frac{2}{c} \sqrt{n \log n}.
\]

The proof of Theorem 2 is based on a result on independent sets in sparse hypergraphs. Given a hypergraph \( H \) on a vertex set \( V \), the sub-hypergraph \( H[X] \) induced by \( X \subset V \) consists of all edges of \( H \) that are contained in \( X \). A hypergraph \( H = (V, E) \) is \( k \)-uniform if every edge \( e \in E \) has size \( k \). Given a \( k \)-uniform hypergraph \( H \) and a set \( S \subset V \) with \( |S| = k - 1 \), the co-degree of \( S \) is the number of all vertices \( v \in V \) such that \( S \cup \{v\} \in E \). Kostochka et al. [4] proved that if \( H \) is a \( k \)-uniform hypergraph, \( k \geq 3 \), with all co-degrees at most \( d \), then \( \alpha(H) \geq c_k \left( \frac{n}{d} \log \frac{n}{d} \right)^{1/d} \), where \( c_k > 0 \).

In fact, a careful look at their proof reveals the following result, that we state for 3-uniform hypergraphs, since this is the case that we need.

Lemma 2.1 ([4]). Let \( H = (V, E) \) be a 3-uniform hypergraph on \( |V| = n \) vertices with all co-degrees at most \( d \), \( d < n/(\log n)^{12} \). Let \( X \) be a random subset of \( V \), obtained by choosing each vertex of \( V \) independently with probability \( p = \frac{n^{-2/5}}{(d\log \log \log n)^{5/4}} \). Let \( Z \) be a set chosen uniformly at random among all the independent sets of \( H[X] \). Then, with high probability \( |Z| = \Omega(\sqrt{n \log n}) \).

With Lemma 2.1 in hand we can now prove Theorem 2.

Proof of Theorem 2: A cell of an arrangement \( A \) is called an \( r \)-cell, if \( r \) lines of \( L \) are forming its boundary. Let \( H_\Delta \subset H_{\text{line-cell}} \) be the 3-uniform hypergraph with the vertex set \( L \) being the set of lines, and each hyperedge defined by the triple of lines forming the boundary of a 3-cell of \( A \). Since any two lines can participate in the boundaries of at most four 3-cells of \( A \), all co-degrees of \( H \) are at most \( d = 4 \). Now, as in Lemma 2.1, let \( X \) be a random subset of \( L \), obtained by choosing each line in \( L \) independently with probability
\[ p = \frac{n^{-2/5}}{(4 \log \log n)^{3/5}}. \] Since there are \( O(n^2) \) faces in \( A \) and \( O(n) \) of them are 2-cells (since every line can bound at most four such faces), the expected number of 2-cells of \( A \) in \( H_{\text{line-cell}}[X] \) is \( O(p^2 n) = o(\sqrt{n \log n}) \), and the expected number of \( r \)-cells, \( r \geq 4 \), of \( A \) in \( H_{\text{line-cell}}[X] \) is \( O(p^4 n^2) = o(\sqrt{n \log n}) \). From Lemma 2.1 it follows that there exists a set \( Z \subset X \subset L \) of size \( \Omega(\sqrt{n \log n}) \), that is an independent set of \( H_{\Delta}[X] \), and such that the number of \( r \)-cells, \( r \neq 3 \), of \( A \) in \( H_{\text{line-cell}}[Z] \) is \( o(\sqrt{n \log n}) \). Removing from \( Z \) one vertex (line) for each such \( r \)-cell, we obtain an independent set of \( H_{\text{line-cell}} \) of size \( \Omega(\sqrt{n \log n}) \).

References


