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The discharging method in combinatorial geometry and the Pach–Sharir conjecture

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Abstract. We review several applications of the *discharging method* in graph theory and in combinatorial geometry. As a new application, we generalize a result of Pach and Sharir about intersection graphs of planar convex sets.

Introduction

The discharging method (DM) is a technique used to prove statements in structural graph theory, and it is commonly applied in the context of planar graphs. It is most well-known for its central role in the proof of the Four Color Theorem, where Heesch's idea of discharging (*Entladung* [**H69b**]) is used to prove that certain configurations are unavoidable in a maximal planar graph (cf. [**AH77**] or later proof in [**R+97**]). Initially, a *charge* of $6-i$ is assigned to each vertex of degree i in a maximal planar graph. Using Euler's formula, it is easy to see that the overall charge is 12. During the *discharging phase*, vertices of positive charge push their charge to other (nearby) vertices (they *discharge*), as required by a set of discharging rules. However, each discharging rule maintains the overall charge. Given that a certain set of configurations $\mathcal{F}$ does not occur, one proves that all vertices can discharge with a nonpositive charge in the end – a contradiction with the overall charge being unchanged and positive; thus, the configurations in $\mathcal{F}$ are unavoidable.

Successful application of DM requires creative design of initial charges and discharging rules. In Section 1, we present a brief survey of numerous existing variants in graph theory. Section 2 shall focus on the recent expanding usage in the realm of combinatorial geometry. In Section 3, we use the DM to make progress towards Turán-type conjecture of Pach and Sharir on the maximum number of edges in H-free intersection graphs of convex sets in the plane.

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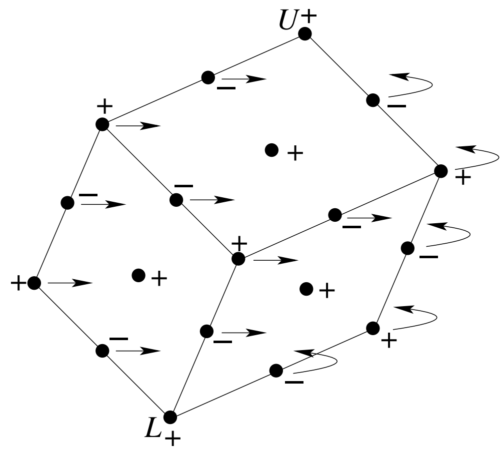

FIGURE 1. Charges for a proof of Euler's formula.

1. Applications of the DM in graph theory

1.1. A "charging scheme" for Euler's formula. Euler's formula is essential in a standard use of DM in order to prove that the sum of initial charges is a small (positive or negative) constant. However, even this simple and classical tool has a DM–type proof, as discovered by Thurston [**T80**]. For a convex polyhedron, we need to prove that the number of vertices and faces together is exactly two more than the number of edges. Arrange the polyhedron in space so that no edge is horizontal; in particular, so there is exactly one uppermost vertex U and lowermost vertex L. Put a $+1$ charge at each vertex, a -1 charge at the center of each edge, and a $+1$ charge in the middle of each face (see Figure 1).

Next, we discharge all the vertex and edge charges into a neighboring face, and then group together all the charges in each face. Each charge moves horizontally, counterclockwise as viewed from above. Each face receives the net charge from an open interval along its boundary, that is decomposed into edges and vertices, which alternate. Since the first and the last are edges, there is a surplus of one -1 charge; therefore, the total charge in each face is zero. All that is left is $+2$, coming from the charges for L and for U.

1.2. Existence of light subgraphs in planar graphs. It is a well known consequence of Euler's formula that every planar graph contains a vertex of degree at most 5. The earliest application of DM dates back to Wernicke [**W04**], who introduced it in 1904 to prove that if a planar triangulation has minimum degree 5, then it contains two adjacent vertices of degree 5 or two adjacent vertices, one of degree 5 and the other of degree 6. For the sake of completeness, we show this simple application of DM in full detail. Pick a plane embedding of this triangulation and use V, F, and E to denote the sets of vertices, faces, and edges, respectively, in the resulting plane graph. Assign a charge of $6-d(v)$ to each vertex v and a charge of $6-2d(f)$ to each face f, where $d(v)$ denotes the degree of a vertex v and $|f|$ denotes the size of a face f, i.e. the number of edges (or vertices) on its boundary.[1]

[1]Since the planar graph is a triangulation, the initial charge on each face is 0.

Since $\sum_{v\in V} d(v) = 2|E|$ and $\sum_{f\in F} |f| = 2|E|$, we have that the overall charge is

$$\sum_{f\in F}(6-2|f|)+\sum_{v\in V}(6-d(v)) = 6|F|-4|E|+6|V|-2|E| = 6(|V|-|E|+|F|) = 12,$$

where the last equality follows from Euler's formula. The only vertices with positive initial charge (equal to 1) are vertices of degree 5. We use a single discharging rule:

- Each vertex of degree 5 gives a charge of $\frac{1}{5}$ to each neighbor.

Clearly, this rule does not change the overall charge, which, in particular, stays positive, so there exists a vertex v with a positive final charge. However, v can only have a positive final charge if $d(v) \leq 7$. If $d(v) = 5$, then v had the initial charge of 1, which it discharged equally among its neighbors; therefore, it had to receive charge from a neighboring vertex u, that had to be of degree 5, in which case we are done. If $d(v) = 6$, then v had the initial charge of 0, so it had to receive charge from a neighboring vertex u, that had to be of degree 5, in which case our proof is again complete. If $d(v) = 7$, then v had the initial charge of -1, so it had to receive charge from at least 6 adjacent vertices of degree 5. Since the graph is a triangulation, two of these neighbors of v must be adjacent.

Wernicke's result was generalized in many directions; namely, it served as a starting point of the quest for *light* subgraphs, i.e. subgraphs of small "weight" in planar graphs, where the weight denotes the sum of vertex degrees [**FJ97, JV**]. The preceeding paragraph shows the existence of a light edge, i.e. an edge with weight at most 11, in every planar graph with minimum degree at least 5. Following some weaker forms of Franklin [**F22**] and Lebesgue [**L40**], Kotzig [**K55**] proved that every 3-connected planar graph[2] contains an edge of weight at most 13, and at most 11 if vertices of degree 3 are absent. In [**FJ97**] it was proved that every 3-connected planar graph containing a path of length k contains such a path with all vertices of degree at most $5k$ (which is best possible); furthermore, the only light subgraphs are paths. Under the additional requirement of minimum degree ≥ 4, paths are still the only light subgraphs [**F+00**], while the minimum degree ≥ 5 already yields existence of many other light subgraphs [**J+99**], whose full characterization is not known. The upcoming survey [**JV**] gives an overview of similar results for various families of plane and projective plane graphs. Recently, Mohar [**M00**], Fabrici et al. [**F07**] studied the existence of light subgraphs in the families of 4-connected planar graphs and 1-planar graphs (graphs that can be drawn in the plane so that every edge is crossed by at most one other edge), respectively.

Many of the theorems mentioned so far fall into the following general framework, as observed in [**M+03, MS04**]: Let W be a list of weight constraints, that is a set of pairs (H, w) where H is a graph and w is an integer. If $\mathcal{G}$ is a class of graphs, let $\mathcal{G}(W)$ be the class of all graphs G from $\mathcal{G}$ such that for every pair $(H, w) \in W$, we have that every subgraph of G isomorphic to H has weight $\geq w$ in G. Now, minimum degree constraints correspond to pairs (K_1, w) in W. A natural question arises: *For a given list of weight constraints W, find all light graphs in $\mathcal{G}(W)$.* Usually, $\mathcal{G}$ is taken to be the class of all planar graphs or some interesting subfamily thereof. Madaras and Škrekovski [**MS04**] go on to study necessary and sufficient conditions for the lightness of certain graphs (paths, stars, cycles) according to values of w in various families of planar graphs and triangulations under edge constraints (K_2, w).

[2]These graphs are edge graphs of polyhedra by Steinitz's theorem.

1.3. Combinatorial structure of neighborhoods in plane graphs. DM–type arguments have also been successfully used in the study of the neighborhood structure of vertices and edges in plane graphs. This direction became apparent after the fundamental paper of Lebesgue in 1940 [**L40**]. One of the stimuli in the subsequent work of other authors were attempts to solve the Four Color Conjecture. Lebesgue coined the term *Euler contribution* and observed that Euler's formula can be equivalently rewritten as

$$\sum_{v\in V}\left(1-\frac{d(v)}{2}+\sum_{f\ni v}\frac{1}{|f|}\right)=2,$$

which implies that there exists a vertex v with a positive contribution: $1-\frac{d(v)}{2}+\sum_{f\ni v}\frac{1}{|f|}>0$. Since $\sum_{f\ni v}\frac{1}{|f|}$ has maximum value $\frac{d(v)}{3}$, one has $\frac{d(v)}{3}>\frac{d(v)}{2}-1$, that is $d(v)<6$. Solving the inequalities for $d(v)\in\{3,4,5\}$, one deduces that in every plane graph there exists: (1) a vertex of degree 3 incident to faces of one of the following sizes[3] $(3,6,?)$, $(3,7,41)$, $(3,8,23)$, $(3,9,17)$, $(3,10,14)$, $(3,11,13)$, $(4,5,19)$, $(4,6,11)$, $(4,7,9)$, $(5,5,9)$, $(5,6,7)$; or (2) a vertex of degree 4 incident to faces of one of the following sizes $(3,3,3,?)$, $(3,3,4,11)$, $(3,3,5,7)$, $(3,4,4,5)$; or (3) a vertex of degree 5 incident to four triangles and a face of size at most 5. Let $w(k)$ denote the minimum weight of a minimum degree vertex in a plane graph, with maximum face size at most $k\geq 3$, where the weight of a vertex is the sum of the sizes of its neighboring faces. Lebesgue's result implies $w(k)\leq\max\{51,k+9\}$. Unifying and strengthening the previous results of Kotzig, Horňak, Jendrol', and others (see [**B95a, B96, HJ96**] and references therein), Borodin and Woodall [**BW98**] used DM to provide exact formulas for w_k. Plane graphs showing optimality of their results correspond to edge graphs of certain 3-polytopes. A detailed description of the edge neighborhoods of 3-connected plane graphs may be found in [**B93**].

1.4. Vertex coloring of planar graphs without prescribed cycles. The problem of deciding whether a planar graph is 3-colorable is NP-complete [**GJ79**]. Therefore, it is natural to discuss sufficient conditions for a planar graph to be 3-colorable. Grötzch proved that planar graphs without 3-cycles are 3-colorable. In 1976, Steinberg conjectured that every planar graph without 4-cycles and 5-cycles is 3-colorable [**JT95**]. This conjecture remains unsettled despite several attempts. Erdős suggested the following relaxation of the problem [**S93**]: does there exist an integer C such that every planar graph without cycles of length between 4 and C is 3-colorable? Abbott and Zhou [**AZ91**] were the first to answer Erdős' question in affirmative, showing that $C\leq 11$. The result has been gradually improved by Sanders and Zhao [**SZ95**] to $C\leq 9$ and by Borodin et al. [**B+05**] to $C\leq 7$. It is now known that if G is a planar graph without 4-, 5-, and 6-cycles, and if it further contains no k-cycles for some fixed $k\in\{7,8,9\}$, then G is 3-colorable (see [**CW07**] and references therein). All the approaches use DM essentially in the same way: first, any plane drawing of a possible minimal counterexample is chosen; then, a set of *reducible configurations* that cannot be present are found; finally, the proofs are completed by the DM, which shows that these configurations are incompatible.

[3] In what follows, "?" denotes a face with no restriction on its size.

The following reenactment of the Euler's formula

$$\sum_{f \in F}(|f|-4) + \sum_{v \in V}(d(v)-4) = -8,$$

is frequently used, while the initial charges are $d(v) - 4$ for each vertex $v \in V$ and $|f| - 4$ for each face $f \in F$, except for the outer face which receives the initial charge of $|f| + 4$ (thus, forcing all the initial charges to sum to 0).

Havel asked in 1969 if there exists a constant C such that every planar graph with minimal distance between 3-cycles at least C was 3-colorable [**H69a**]. The (strong) Bordeaux conjecture is sort of an "intersection" of Steinberg's and Havel's problem and it states that every planar graph without adjacent (intersecting) 3-cycles and without 5-cycles is 3-colorable.[4] It is only known that every planar graph with neither 3-cycles at distance ≤ 3 nor 5-cycles is 3-colorable [**BR03**], where the DM was a major ingredient again. For choosability analogues consult [**M+06**].

Another classical conjecture on vertex coloring of planar graphs is due to Wegner [**JT95**]: the chromatic number of the square of a planar graph $\chi(G^2)$ is at most $\lfloor \frac{3}{2}\Delta \rfloor + 1$, if the maximum vertex degree Δ is at least 8; and at most $\Delta + 5$, if $4 \leq \Delta \leq 7$. These bounds would be best possible. Improving several previous results, Molloy and Salavatipour [**MS05**] use the DM to prove that $\chi(G^2) \leq \lfloor \frac{5}{3}\Delta \rfloor + O(1)$. They also study $L(p,q)$-labelings in connection with the frequency assignment problems in radio and cellular phone systems.

1.5. Cyclic and acyclic colorings of the vertices of plane graphs. One of the most fruitful applications of the DM has been in the study of several coloring parameters of plane graphs, other than the chromatic number. Here, we survey only a fraction of the rich literature on this subject.

A *cyclic coloring* of a plane graph is a coloring of its vertices such that any two distinct vertices incident with the same face receive distinct colors. Clearly, the number of colors used has to be at least the size Δ^* of the largest face of a plane graph. Let $\chi_c(\Delta^*)$ be the minimum number of colors needed in a cyclic coloring of every plane graphs with maximum face size $\leq \Delta^*$. The best known lower bound of $\lfloor \frac{3}{2}\Delta^* \rfloor$ is also conjectured to be the best possible (see [**JT95**], p. 37). Ore and Plummer proved the first upper bound $\chi_c(\Delta^*) \leq 2\Delta^*$ in [**OP69**]. After several gradual improvements, Sanders and Zhao [**SZ02a**] proved the best known upper bound $\chi_c(\Delta^*) \leq \lfloor \frac{5}{3}\Delta^* \rfloor$. Better results are known for small values of Δ^*. The Four Color Theorem can be restated as $\chi_c(3) = 4$. The case of $\Delta^* = 4$ was Ringel's conjecture, resolved by Borodin (c.f. [**B95b**]). Some partial results for $\Delta^* \in \{5, 6, 7\}$ can be found in [**B+07**] and references therein.

For 3-connected plane graphs (i.e. 1-skeleta of 3-polytopes), there was a conjecture of Plummer and Toft that $\chi_c(\Delta^*) \leq \Delta^* + 2$, whenever $\Delta^* \geq 3$. Horňak and Jendrol' [**HJ99**] confirmed the conjecture for $\Delta^* \geq 24$. Furthermore, by finding appropriate reducible configurations and using clever discharging rules, Enomoto et al. [**E+01**] prove that $\chi_c(\Delta^*) \leq \Delta^* + 1$, with $\Delta^* \geq 60$, improving previous results by Borodin and Woodall [**BW99**]. At present, the sharp upper bound on $\chi_c(\Delta^*)$ for 3-connected plane graphs remains unknown whenever $5 \leq \Delta^* \leq 59$. *Facial*, *diagonal* and *distance* colorings of plane graphs are natural generalizations of the cyclic coloring (see e.g. Problems 2.15 and 3.10 in [**JT95**]). Historical background

[4] Adjacent (intersecting) 3-cycles are triangles with an edge (a vertex) in common.

and examples of applications of DM in the study of these coloring variants can be found in [**SZ02a, K+05, MR06, H+07**].

A proper vertex coloring of a graph is *acyclic* if every cycle uses at least three colors. These colorings were introduced by Grünbaum [**G73**], who proved that every planar graph is acyclically 9-colorable. The result was steadily improved (see [**D+05**] and references therein) until Borodin [**B79**] used the DM and proved that every planar graph is acyclically 5-colorable, which is the best possible bound. In [**B+99b**], it is proved that every planar graph with girth ≥ 5 (resp. ≥ 7) is acyclically 4-colorable (resp. 3-colorable). Many other graph families have bounded acyclic chromatic number; here we mention only those for which the property was proved using the DM: graphs embeddable on a fixed surface [**A+96, B+02b**], and 1-planar graphs [**B+99a**]. Recently, Dujmović et al. [**D+04, D+05**] discovered connections between the acyclic chromatic number and track and queue layouts of graphs, which in turn have ramifications in graph drawing.

The concept of acyclic coloring has been successfully extended to the question of acyclic choosability of planar graphs. In [**B+02a**], it is proved that every planar graph is acyclically 7-choosable, that is, if each vertex v of a planar graph G has a list $L(v)$ of at least 7 admissible colors, then one can choose a color from $L(v)$, so that the resulting coloring of G is acyclic. The proof is based on a structural theorem, that states a sufficient condition for a plane triangulation to have a face of weight at most 17, and is proved by DM–type arguments (also see [**B89**] for history of Kotzig's conjecture). Since Thomassen proved that each planar graph is 5-choosable, it seems wise to suspect that every planar graph is acyclicaly 5-choosable. This is only known for planar graphs without 4-cycles and without 5-cycles (or 6-cycles) [**M+07**].

1.6. Simultaneous colorings of plane graphs. For convenience, in this section, the term *adjacent* will replace the two standard terms of adjacent and incident. A great amount of interest and successful applications of DM has been devoted to the study of the problem of simultaneous colorings [**F71, J69**], that is, colorings of some or all of the elements (vertices, edges, and faces) of plane graphs so that distinct adjacent elements receive different colors [**JT95**]. One usually considers the minimum number of colors needed in such a coloring for plane graphs of maximum degree Δ, giving rise to chromatic numbers $\chi_v(\Delta)$, $\chi_e(\Delta)$, $\chi_f(\Delta)$, $\chi_{vf}(\Delta)$, $\chi_{ve}(\Delta)$, $\chi_{ef}(\Delta)$, and $\chi_{vef}(\Delta)$. The Four Color Theorem [**AH77, R+97**] then states that $\chi_v(\Delta) \leq 4$ (and hence $\chi_f(\Delta) \leq 4$), while Vizing's theorem (that generalizes to all graphs, not just planar ones) states that $\chi_e(\Delta) \leq \Delta + 1$ [**V64**]. Borodin's resolution of Ringel's conjecture (already mentioned in the previous section) is equivalent to $\chi_{vf}(\Delta) \leq 6$ [**B95b**]. None of the chromatic numbers involving edge colorings has been determined precisely. For instance, although $\chi_e(\Delta) \leq \Delta + 1$ cannot be improved for all Δ (in particular, $\Delta \in \{2, 3, 4, 5\}$), Vizing [**V68**] proved that $\chi_e(\Delta) = \Delta$ for $\Delta \geq 8$ and conjectured that the same holds for $\Delta \in \{6, 7\}$. Zhang [**Z00**] used the DM and proved Vizing's conjecture for $\Delta = 7$ (also c.f. [**SZ01b**]), while the case $\Delta = 6$ remains open (see [**BW06**]). Using the DM, Sanders and Zhao [**SZ01a**] proved Melnikov's conjecture [**M75**] that $\chi_{ef}(\Delta) \leq \Delta + 3$. The currently best results on $\chi_{ef}(\Delta)$ can be summarized as follows: $\chi_{ef}(2) = 5$, $\chi_{ef}(3) \leq 5$, $\chi_{ef}(\Delta) \leq \Delta + 3$ for $\Delta \in \{4, 5, 6\}$, $\chi_{ef}(\Delta) \leq \Delta + 2$ for $\Delta \in \{7, 8, 9\}$, and $\chi_{ef}(\Delta) = \Delta + 1$ for $\Delta \geq 10$ [**B94, SZ01a**]. For 2-connected plane graphs it is known that $\chi_{ef}(\Delta) = \Delta$ for $\Delta \geq 24$ [**LZ05**]. Analogues for surfaces of higher genus are resolved in [**SZ03, L+06**] using DM–type arguments, while the choosability extensions can be found in [**WL04, C**] and the reference therein. The remaining two simultaneous coloring problems are unsolved, but most cases have been completed, again using clever discharging rules. Vizing's conjecture [**V64**] (also for general graphs) that $\chi_{ve}(\Delta) \leq \Delta + 2$ is only open for $\Delta = 6$ [**SZ99**]. Kronk and Mitchem's conjecture [**KM72**] that $\chi_{vef}(\Delta) \leq \Delta + 4$ is only open for $\Delta \in \{4, 5\}$ [**SZ00**].

1.7. Edge chromatic critical graphs. By Vizing's theorem, mentioned in the previous section, the edge chromatic number $\chi_e(\Delta)$ of a graph G (not necessarily planar) of maximum degree Δ is either Δ or $\Delta + 1$. If G is a connected graph of maximum degree Δ such that $\chi_e(\Delta) = \Delta + 1$, but $\chi_e(G \setminus e) < \chi_e(G)$ for every edge $e \in E(G)$, then G is said to be Δ-critical. It was conjectured by Vizing [**V68**] that if G is a Δ-critical graph then the number of edges $e(G)$ is at least $\frac{1}{2}(n(\Delta - 1) + 3)$. The conjecture has been verified for $\Delta \leq 5$. The best known lower bounds on $e(G)$ were around $\frac{1}{4}n(\Delta + 1)$, until Sanders and Zhao [**SZ02b**] used the DM to show that $e(G) \geq \frac{f(\Delta)n}{2}$, where $f(\Delta) = \frac{1}{2}(\Delta + \sqrt{2\Delta - 1})$. Refining the charging rules, Zhao [**Z04**] was able to obtain the best lower bounds on $e(G)$ for $\Delta \in \{6, \ldots, 11\}$. It is instructive to see how the DM is applied here, since one considers general graphs, and hence Euler's formula is not applicable. Suppose there exists a Δ-critical graph $G = (V, E)$ with $|E| < \frac{1}{2}f(\Delta)|V|$. The essential tool here is Vizing's adjacency lemma which states that for every vertex of a Δ-critical graph with at least one neighbor of degree i, the number of neighbors of degree Δ is at least $\max\{2, \Delta - i + 1\}$. For each vertex $v \in V$, define the initial charge $\mathrm{ch}(v) = f(\Delta) - d(v)$. Then, $\sum_{v \in V} \mathrm{ch}(v) = f(\Delta)|V| - 2|E| > 0$. Next, one assigns a new charge denoted by $\mathrm{ch}_1(v)$ to each $v \in V$ according to the single discharging rule:

- Let v be a vertex of degree less than $f(\Delta)$. Then v discharges $\frac{d(u) - f(\Delta)}{d(u) + d(v) - \Delta - 1}$ to each adjacent vertex u of degree greater than $f(\Delta)$.

Now, it is not difficult to show that $\sum_{v \in V} \mathrm{ch}(v) = \sum_{v \in V} \mathrm{ch}_1(v)$ and, eventually, that $\mathrm{ch}_1(v) \leq 0$ for each $v \in V$, which is a contradiction.

We have only touched the tip of the iceberg in terms of the applications of the DM in graph theory. For other examples of clever discharging arguments, see [**A+05, B+04, B07, CK07, HI02, S06, SZ01c, SW04, VW02, Z03**].

2. Applications of the DM in discrete geometry

Euler's polyhedral formula appears in many disguises throughout combinatorial geometry literature. One of the typical occurrences is in connection with Sylvester-Gallai–type problems (c.f. [**CS93**]). Given an arrangement of circles (or lines) in the plane, Euler's formula is applied to the plane graph obtained by introducing a vertex at each intersection point, and considering the segments of the curves as edges. It is usually restated as

$$\sum_{k \geq 2} (k - 3)t_k + \sum_{k \geq 2} (k - 3)f_k = -6,$$

where t_k denotes the number of intersection points of exactly k circles (lines), and f_k the number of faces of size k (each receiving $k - 3$ as the initial charge, -6 in total). Notably, in [**P02**], Pinchasi proved a conjecture of A. Bezdek that every finite family of at least five pairwise intersecting unit circles in the plane contains an intersection point that lies on exactly two circles. In [**PP00**], Pach and Pinchasi proved weak version of Fukuda's conjecture regarding the existence of a bichromatic line through at most two blue and at most two red points in any given set of n blue and n red points in the plane, not all on a line. Although their proofs are actually clever counting arguments combined with the above identity, they can be paraphrased using the discharging methodology.

In [**P07**], Euler's formula in the above form is used again to prove that every set of n points in the plane, not all on a line, determines at least $\lfloor \frac{n-1}{2} \rfloor$ triangles with pairwise distinct areas, hence confirming an old conjecture of Erdős, Purdy, and Straus. A set P of points is called *magic*, if there is an assignment of positive weights to the points of P so that for every line ℓ determined by P, the sum of the weights of all points of P on ℓ is the same. In [**A+06**], a delicate discharging scheme is executed in order to prove an old conjecture of U. S. R. Murty that, if P is a magic configuration then either (1) P is in general position, or (2) P contains $|P|-1$ collinear points, or (3) P is the "failed Fano configuration"[5]. In [**P+06a**], Pach et al. studied the lower bound on the crossing number $\mathrm{cr}(G)$ of an arbitrary graph G, that is, the minimum number of edge crossings in a drawing of G in the plane (under some natural restrictions). In the seminal papers in the early 80's, Ajtai, Chvátal, Newborn, Szemerédi and, independently, Leighton discovered that for every graph G with $E(G)$ edges and $V(G)$ vertices, cr(G) is at least $C|E(G)|^3/|V(G)|^2$, where $C > 0$ is an absolute constant. This result, known as the "Crossing Lemma", has found many important applications in discrete geometry, number theory, and VLSI design, and it is tight up to a multiplicative constant C, whose exact value is difficult to find and has attracted a lot of attention. All the known proofs of the Crossing Lemma are based on the immediate corollary $\mathrm{cr}(H) \geq |E(H)| - (3|V(H)| - 6)$ of Euler's formula, which is subsequently applied to small and mostly sparse subgraphs H of G, or to a randomly selected one. After several gradual improvements, the best known lower bound $C > \frac{1024}{31827} > 0.032$ is established in [**P+06a**] via a stronger inequality for the sparse subgraphs. Namely, using the DM, the authors show that every 3-planar graph (a graph that can be drawn in the plane so that every edge crosses at most three others) has at most $5.5(|V(G)| - 2)$ edges. The DM has been successfully applied in the study of crossing-critical graphs as well. It is well known that for every positive integer k, there is a graph G and an edge e of G such that $\mathrm{cr}(G) = k$, but $G - e$ is planar. In [**RT93**], Richter and Thomassen conjectured that there is a constant c such that for every graph G, there is an edge e such that $\mathrm{cr}(G - e) \geq \mathrm{cr}(G) - c\sqrt{\mathrm{cr}(G)}$. They only showed that G always has an edge e with $\mathrm{cr}(G - e) \geq \frac{2}{5}\mathrm{cr}(G) - O(1)$, which was improved by Salazar [**S00**] to $\mathrm{cr}(G - e) \geq \frac{1}{2}\mathrm{cr}(G) - O(1)$ in the case when G has no vertices of degree 3 (c.f. [**LS06, FT06**]). Their approach uses DM to find "nearly light" cycles, i.e. short cycles with at most one vertex of high degree, in embedded graphs; and is very reminiscent of Lebesgue's theory of Euler contributions from Section 1.3.

Discharging schemes among vertices in triangulations, that rely on the structure of the set of all triangulations imposed by edge flips, were used by Sharir and Welzl in [**SW06**] to investigate the expected number $\hat{v}_i$ of interior points of degree i in a triangulation of a finite set P of $n+3$ points in general position in the plane (with

[5] Up to a projective transformation a "failed Fano configuration" is the three vertices of a triangle, the midpoints of the sides, and the centroid.

3 extreme points and n interior points), that is drawn uniformly at random from all triangulations of P. They proved that $n/43 \leq \hat{v}_3 \leq (2n+3)/5$, and proceeded to use it to provide the best known upper bound of 43^n on the maximum possible number of triangulations of any set of n points in the plane. Their DM approach significantly differs from the previous ones. First, they let every vertex have the initial charge of $7-i$. This way they make sure the overall charge in a maximal planar graph is at least n, or equivalently, there is at least one unit of charge per every vertex on the average. Second, their discharging rules are applied across a family of all triangulations of the given set, with charge going from a vertex v in one triangulation T to vertices in the triangulations obtained by flipping a single edge incident to v in T. The charge is redistributed so that no vertex of degree exceeding 3 has positive charge, while the vertices of degree 3 have charge at most 43. This implies that at least 1/43 of all vertices over all triangulations have degree 3.

Another interesting class of problems where the DM has bore fruit is the estimation of the chromatic number of intersection graphs in the plane, which was initiated by Asplund and Grünbaum (see e.g. [**KN98, K04**]). In [**KP00**], Kostochka and Perepelitsa show that every intersection graph of axis-parallel rectangles with girth at least 6 (or 8) is $4-$(or $3-$) colorable.

2.1. Extremal questions for quasi-planar graphs. One of the most recent and fundamental contributions of the DM is in the extremal theory of geometric graphs [**P91**]. A *geometric graph* G is a graph drawn in the plane, that is, its vertex set, $V(G)$ is a set of distinct points, and its edge set, $E(G)$, is a set of straight line segments, each connecting two vertices and containing no other vertex.[6] It is typically assumed that no three edges of G cross in a single vertex. A geometric graph is *k-quasi-planar* if no k of its edges are pairwise crossing. It is a folklore conjecture [**P91**] that the maximum number of edges, $f_k(n)$, in a k-quasi-planar graph on n vertices is at most $c_k n$, where the constant c_k depends on k. For $k=2$, it is immediate from Euler's formula that $f_2(n) = 3n-6$. There are several proofs that $f_3(n) = O(n)$ [**A+97, P+06b**], but the most recent one, due to Ackerman and Tardos [**AT07**], uses the DM and provides the best value of the constant $c_3 = 8$ (which is very close to best lower bound construction with $7n - O(1)$ edges). Ackerman [**A06**] went on to prove that $f_4(n) \leq 36n - O(1)$, again using the DM approach and some additional ideas. The best upper bound for $k \geq 5$ is $O(n \log^{4k-16} n)$.

Since it is very instructive for the presentation of our results later on in the paper, we shall describe Ackerman's original approach for $k=3$ in some detail here. We will not care about the best value of the constant c_3 and will prove $f_3(n) \leq 10n - 20$ instead. Consider a 3-quasi-planar graph G on n vertices. Let $\tilde{G}$ be the planar graph together with its planar drawing, induced by $E(G)$; that is, the vertices $V(\tilde{G})$ are the endpoints of the segments (called *end-vertices*) and the crossings of the segments (called *crossing-vertices*). The edges of G are subdivided into the edges of $\tilde{G}$ accordingly (see Figure 2). Let $V_e(\tilde{G})$ (resp. $V_c(\tilde{G})$) denote the set of end-vertices (resp. crossing-vertices) of $\tilde{G}$, and let $E(\tilde{G})$ (resp. $F(\tilde{G})$) denote

[6] Our discussion in this section also applies to more general *topological graphs* [**P04**], in which the edges may be drawn with non-self-intersecting Jordan arcs; however, this is not important for the present paper.

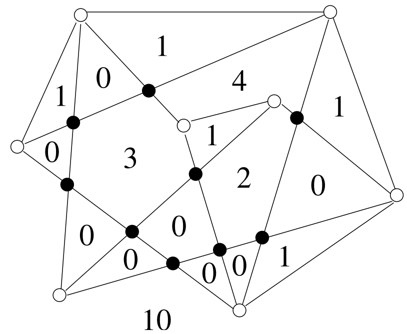

FIGURE 2. End-vertices (in white), crossing-vertices (in black), and initial charges.

the edges (resp. faces) of $\tilde{G}$. Note that as no three edges of G cross in a single vertex, all the crossing-vertices have degree 4. For each face f, let $|f|$ denote the number of edges along its boundary, and let $v_e(f)$ denote the number of end-vertices on its boundary. Note that an edge of $\tilde{G}$ may appear twice along the boundary of a face. A face f will be shortly called a $v_e(f)$–$|f|$-gon. For instance, a 1-triangle denotes a face of size 3 with 1 end-vertex.

Initial charges $\mathrm{ch}(f) = |f| + v_e(f) - 4$ are assigned to the faces of $\tilde{G}$ (see Figure 2). The overall charge is

$$\sum_{f \in F(\tilde{G})} \mathrm{ch}(f) = \sum_{f \in F(\tilde{G})} (|f| + v_e(f) - 4) = 2|E(\tilde{G})| + \sum_{f \in F(\tilde{G})} v_e(f) - 4|F(\tilde{G})| = 4n - 8,$$

where we used Euler's formula and the obvious identity

$$\sum_{f \in F(\tilde{G})} v_e(f) = \sum_{u \in V_e(\tilde{G})} d(u) = \sum_{u \in V(\tilde{G})} d(u) - \sum_{u \in V_c(\tilde{G})} d(u)$$
$$= 2|E(\tilde{G})| - 4(|V(\tilde{G})| - |V(G)|),$$

where $d(u)$ denotes the degree of u.

Graph $\tilde{G}$ does not have faces of size 1 or 2. Furthermore, there are no 0-triangles, since G is 3-quasi-planar. Note that every face in $\tilde{G}$ has a non-negative initial charge. Next, we redistribute the charges without affecting the total charge of $4n-8$, while making sure that the new charge $\mathrm{ch}_1(f)$ of a face f satisfies $\mathrm{ch}_1(f) \geq v_e(f)/5$. Note that the only faces which do not already have enough charge are 1-triangles. The idea is to "walk" along the wedge formed by the original vertex of a 1-triangle and the two incident edges, and eventually find a face with plenty of charge to discharge. More precisely, let $f := f_0$ be a 1-triangle, and let e_1 be the edge not incident to its original vertex. Let f_1 be the other face incident to e_1 (see Figure 3). If $v_e(f_1) > 0$ or $|f_1| > 4$, f_1 discharges $1/5$ unit of charge through e_1 (which is then called an *active* edge) to f. Otherwise, f_1 must be a 0-quadrilateral. Let e_2 be an edge of $\tilde{G}$ opposite to e_1 in f_1, and let f_2 be the other face incident to e_2. Applying the same argument as above, we conclude that either f_2 discharges $1/5$ unit of charge through e_2 to f, or f_2 is also a 0-quadrilateral. In the second case, we continue to the next face in the wedge. At some point, we must encounter a face f_i with $v_e(f_1) > 0$ or $|f_1| > 4$, in particular, f_i is not a 0-quadrilateral. Then, f_i discharges $1/5$ unit of charge through e_i to f.

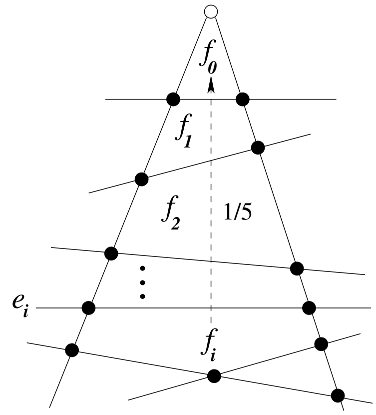

FIGURE 3. Walking along the wedge and charging a 1-triangle.

Observe that $\mathrm{ch}_1(f) = 1/5 = v_e(f)/5$ for every 1-triangle f and $\mathrm{ch}_1(f) = 0 = v_e(f)/5$ for every 0-quadrilateral f. Now, let f be a face of $\tilde{G}$ that is neither a 0-quadrilateral nor a 1-triangle. We have $\mathrm{ch}(f) = |f| + v_e(f) - 4 \geq 1$ and $\mathrm{ch}_1(f) = \mathrm{ch}(f) - c_f$, where c_f is the charge that f lost in the discharging phase. Since an edge of f is active only if both endpoints of the edge are new vertices, we have $c_f \leq (|f| - v_e(f))/5$; hence, $\mathrm{ch}_1(f) \geq (2/5)v_e(f) + (4/5)(\mathrm{ch}(f) - 1) \geq (2/5)v_e(f)$. Therefore, we have $\mathrm{ch}_1(f) \geq v_e(f)/5$ for all faces f of $\tilde{G}$. Finally, we obtain

$$|E(G)| = \frac{1}{2}\sum_{u\in V(G)} d(u) = \frac{1}{2}\sum_{f\in F(\tilde{G})} v_e(f) \leq \frac{5}{2}\sum_{f\in F(\tilde{G})} \mathrm{ch}_1(f) = \frac{5}{2}\sum_{f\in F(\tilde{G})} \mathrm{ch}(f)$$
$$= \frac{5}{2}(4n-8) = 10n - 20.$$

3. On planar intersection graphs with forbidden subgraphs

Given a collection $\mathcal{C} = \{C_1, \ldots, C_n\}$ of compact connected sets in the plane, their intersection graph $G(\mathcal{C})$ is a graph whose vertices correspond to the sets, and two vertices are connected if the corresponding sets intersect. For any graph H, a graph G is called H-*free* if it does not contain a subgraph isomorphic to H. Pach and Sharir [**PS07**] started investigating the maximum number of edges an H-free intersection graph $G(\mathcal{C})$ on n vertices can have. If H is not bipartite, then the assumption that G is an intersection graph of compact connected sets in the plane does not effect the answer. Namely, according to the Erdős-Stone theorem, we have that the maximum number of edges in an H-free graph on n vertices is given by

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\frac{n^2}{2},$$

where $\chi(H)$ is the chromatic number of H. This bound is asymptotically tight if H is not bipartite, as it can be shown by Turán's complete $(\chi(H) - 1)$-partite graph whose vertex classes are of roughly equal size. This graph, in turn, can be realized geometrically as the intersection graph of a collection of segments in the plane, where the segments in each of the vertex classes are parallel.

The problem becomes much more interesting if H is bipartite. The classical theorem of Kővári, Sós, and Turán provides a subquadratic bound on the number edges in $G(\mathcal{C})$, since $\mathrm{ex}(n, K_{k,k}) = O(n^{2-1/k})$. However, geometry does make the difference here, as Pach and Sharir were able to prove that for every positive integer k there is a constant c_k such that, if $G(\mathcal{C})$ is a $K_{k,k}$-free intersection graph of n convex sets in the plane, then it has at most $c_k n \log n$ edges. In other words, for every collection of n convex sets in the plane with no k of them intersecting k others, there are at most $c_k n \log n$ intersecting pairs. In the case $k = 2$, they reduced their bound by a $\log n$ factor to $O(n)$. They further conjectured the following.

Conjecture 3.1 (Pach–Sharir). Given a bipartite graph H, there exists a constant c_H such that every H-free intersection graph of n convex sets in the plane has at most $c_H n$ edges.

Here we prove their conjecture for $H \in \{K_{2,3}, C_6\}$. Our proof is based on the ideas of Pach and Sharir, and the DM-type argument of Ackerman [**A06**].

Theorem 3.2. *Suppose that the intersection graph of n convex sets in the plane does not contain*

(i): $K_{2,3}$
(ii): C_6

as a subgraph. Then its number of edges is $O(n)$.

Proof. Let $\mathcal{C}$ be a collection of n convex sets in the plane such that their intersection graph does not contain (i) $K_{2,3}$, or it does not contain (ii) C_6 as a subgraph. Add four more sets to $\mathcal{C}$, all four being very long horizontal segments, two of them above all original sets in $\mathcal{C}$, and two of them below all original sets in $\mathcal{C}$. Now we have $n+4$ sets, their intersection graph has the same number of edges, and it still does not contain $K_{2,3}$ or C_6, respectively, as a subgraph.

In both cases, it follows that the intersection graph does not contain $K_{3,3}$ as a subgraph. In the first part of the proof we only use this condition. We follow the ideas of Pach and Sharir. For any $C \in \mathcal{C}$ let s_C denote the *spine* of C, the segment connecting its leftmost and rightmost points. Let $\mathcal{S}$ denote the set of spines and $\mathcal{A}(\mathcal{S})$ denote their arrangement. Apply a little perturbation to the sets, if necessary, so that their intersection graph remains the same, but the spines are in general position, that is, no three spines cross at the same point, and no three endpoints are collinear.

Let Ξ denote the vertical decomposition of the arrangement $\mathcal{A}(\mathcal{S})$ of the spines. That is, erect a vertical segment up and down from each endpoint and from each intersection of the segments, until they hit another segment, or else all the way to infinity. Each cell of Ξ is a trapezoid, bounded by (portions of) the spines on the top and bottom, and by vertical segments on the left and on the right; any of these boundary segments may be missing. Let X denote the number of intersections of the spines.

Let Δ be a cell of Ξ, A and B be two of the sets such that s_A (resp. s_B) contains the upper (resp. lower) boundary of Δ. Let $K \in \mathcal{C}$, $K \neq A, B$, such that K intersects Δ, and let p be a point in the intersection. Let λ be the vertical line through p. Order all spines that intersect λ according to the order of the intersections. Clearly, s_A is above s_B and they are neighbors in this order.

Assume that s_K is *below* s_B. Suppose that there are at least two other spines between s_B and s_K. Let s_C be the spine immediately below s_B, and let s_D be

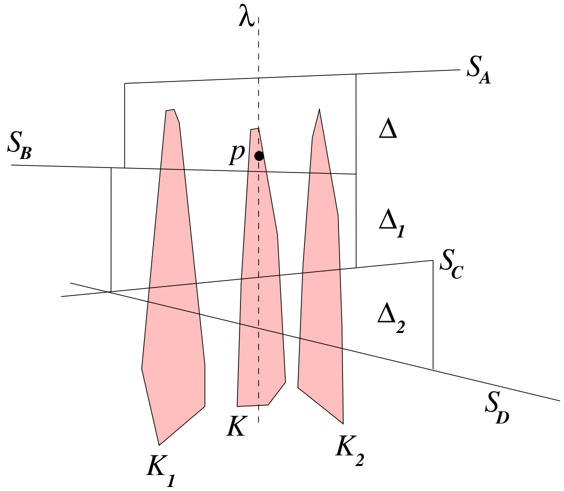

FIGURE 4. K, K_1, and K_2 are 1-assigned to $(\Delta, \Delta_1, \Delta_2)$, thus forming a $K_{3,3}$ with B, C, and D in $G(\mathcal{C})$.

immediately below s_C. Let Δ_1 be the cell below Δ along line λ and let Δ_2 be the cell below Δ_1 along λ. We say that K is 1–assigned to the ordered triple $(\Delta, \Delta_1, \Delta_2)$. We claim that there is at most one other set 1–assigned to $(\Delta, \Delta_1, \Delta_2)$. Indeed, suppose that K, K_1, and K_2 are all 1–*assigned* to $(\Delta, \Delta_1, \Delta_2)$. Then each of K, K_1, and K_2 must intersect each of B, C, and D, forming a forbidden $K_{3,3}$ in the intersection graph of $\mathcal{C}$ (see Figure 4).

Analogously, if s_K is *above* s_A, then s_K is either the neighbor, or the second neighbor above s_A, or it is 2–*assigned* to the triple $(\Delta, \Delta_3, \Delta_4)$, where Δ_3 is the cell above Δ along line λ, and Δ_4 is the cell above Δ_3 along λ. We can observe again that there are at most two sets 2–assigned to $(\Delta, \Delta_3, \Delta_4)$.

Now let ℓ be a vertical line and suppose that $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5$ are five consecutive cells along λ from top to bottom. We say that $(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5)$ is a *good quintuple*, and $K \in \mathcal{C}$ is *assigned* to it if either (i) K is 1–assigned to $(\Delta_3, \Delta_4, \Delta_5)$, or (ii) K is 2–assigned to $(\Delta_3, \Delta_2, \Delta_1)$, or (iii) s_K contains the upper or lower boundary of any of $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5$.

Let us estimate the number of good quintuples. Sweep Ξ by a vertical line ℓ from left to right and maintain the list of all good quintuples that intersect λ. Initially, when λ is very far to the left, we have no such good quintuples. The list changes when we pass through an endpoint of a segment or an intersection of two segments, and in each case we get at most five new good quintuples. Therefore, we have at most $10n + 5X + 40$ good quintuples.[7]

Observe that if two sets A and B intersect each other, then they are both assigned to the same good quintuple. By the previous argument, there are at most 10 sets of $\mathcal{C}$ that are assigned to the same good quintuple, and they contribute at most $\binom{10}{2} = 45$ pairwise intersections. It follows that the intersection graph of $\mathcal{C}$ has at most $450n + 225X + 1800$ edges.

[7]Four long horizontal segments that we have added to $\mathcal{C}$ at the beginning of our proof prevent any degenerate quintuples.

Therefore, it remains to show that $X = O(n)$; more precisely, we have to prove the following statement (in Case (ii) we only sketch the argument). $\square$

Lemma 3.3. *Let $\mathcal{S}$ be a collection of n segments in the plane such that their intersection graph $G(\mathcal{S})$ does not contain*

(i): $K_{2,3}$
(ii): C_6

as a subgraph. Then $G(\mathcal{S})$ has $O(n)$ edges.

Proof. **Case (i):** $G(\mathcal{S})$ does not contain $K_{2,3}$ as a subgraph. As in Section 2.1, let $\tilde{G}$ be the planar graph together with its planar drawing, induced by $\mathcal{S}$. We use the same notation: $V_e(\tilde{G})$, $V_c(\tilde{G})$, $E(\tilde{G})$, $F(\tilde{G})$, $|f|$, $v_e(f)$. Note that in our case $|V_e(\tilde{G})| = 2n$, and any face f with $|v_e(f)| > 0$ has size $|f| \geq 6$. Following [**A06**], assign initial charge $\mathrm{ch}(f) = |f| + v_e(f) - 4$ to each face f. For the total charge we have

$$\sum_{f\in F(\tilde{G})} \mathrm{ch}(f) = \sum_{f\in F(\tilde{G})} (|f| + v_e(f) - 4) = 2|E(\tilde{G})| + \sum_{f\in F(\tilde{G})} v_e(f) - 4|F(\tilde{G})|$$

$$= 2|E(\tilde{G})| + 2n - 4|F(\tilde{G})| = 2|E(\tilde{G})| + 2n + 4|V_e(\tilde{G})| + 4|V_c(\tilde{G})| - 4|E(\tilde{G})| - 8$$
$$= -2|E(\tilde{G})| + 10n + 4|V_c(\tilde{G})| - 8 = 8n - 8.$$

We used that

$$|E(\tilde{G})| = n + 2|V_c(\tilde{G})|,$$

as well as Euler's formula for $\tilde{G}$.

The only faces with negative charge are 0-triangles, and the only faces with 0 charge are the 0-quadrilaterals. We redistribute the charges so that all faces have nonnegative charges. Then we move the charges from the faces to the crossing-vertices on their boundary, and this will imply a linear upper bound on the number of crossings. Our charge redistribution is very reminiscent of Ackerman's; here, instead of discharging 1/5 to a 1-triangle from one direction, we discharge 1/3 to a 0-triangle from all three directions.

Charge Redistribution. Let t be a 0-triangle, and e_1, e_2, e_3 its edges. Let f_1 be the other face incident to e_3. It cannot be a 0-triangle since e_1 and e_2 cannot intersect twice. If $|f_1| > 4$, then move 1/3 charge from f_1 to t. Otherwise, f_1 is a 0-quadrilateral. Let e_4 be the side of f_1 opposite to e_3, and consider the face f_2 on the other side of e_4. Just like in the previous case, f_2 cannot be a 0-triangle. If $|f_2| > 4$, then move 1/3 charge from f_2 to t, and if f_2 is a 0-quadrilateral, then consider the next face f_3. Proceed analogously in this fashion, and at some point we have to encounter a face f with $|f| > 4$. Then move 1/3 charge from f to t. Do the same for all 0-triangles, in all three directions (see Figure 5). Let $\mathrm{ch}_1(f)$ denote the *modified charge* of a face f.

Claim 3.4. We have $\mathrm{ch}_1(f) = 0$ if and only if f is a 0-triangle or a 0-quadrilateral. Otherwise, $\mathrm{ch}_1(f) \geq 2|f|/21$.

Proof. It is clear that $\mathrm{ch}_1(f) = 0$ for 0-triangles and 0-quadrilaterals. We show that $\mathrm{ch}_1(f) \geq 2|f|/21$ for all other types of faces. Any face f gives 1/3 charge to at most $|f|$ triangles; therefore, if $|f| \geq 7$, then $\mathrm{ch}_1(f) \geq |f| - 4 - |f|/3 \geq 2|f|/3 - 4|f|/7 \geq 2|f|/21$.

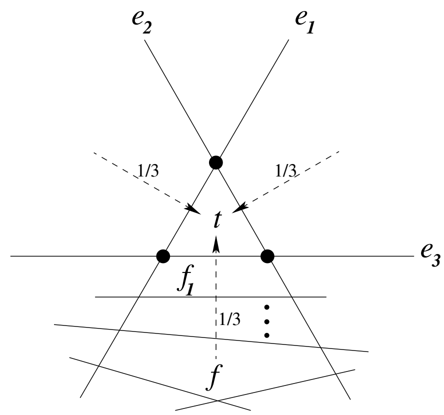

FIGURE 5. Discharging to 0-triangles.

Suppose that $|f| = 6$. Then f is either a 0-hexagon, or a 1-hexagon. If it is a 1-hexagon, then $\mathrm{ch}(f) = 3$, so $\mathrm{ch}_1(f) \geq 1 > 2|f|/21$. If f is a 0-hexagon, let $s_1, \ldots, s_6$ denote the segments containing the sides of f, in counterclockwise direction. If f gave charge to at most five 0-triangles then $\mathrm{ch}_1(f) \geq 1/3$ and we are done. If it gave charge to six 0-triangles, then each segment s_i crosses s_{i-1} and s_{i+1} (indices are taken mod 6), so their intersection graph contains a $K_{2,3}$ (see Figure 6A).

Now suppose that $|f| = 5$. Then f is a 0-pentagon, $\mathrm{ch}(f) = 1$, and just like in the previous argument, it is not hard to see that f could give charge to at most two 0-triangles (see Figure 6BC), so $\mathrm{ch}_1(f) \geq 1/3$ and we are done. □

Now we do the second redistribution of the charges.

CHARGING CROSSINGS. For each face with $\mathrm{ch}_1(f) > 0$, we know that $\mathrm{ch}_1(f) \geq 2|f|/21$. Move 2/21 charge to each crossing-vertex on its boundary.

CLAIM 3.5. **(i):** Each crossing-vertex gets charge at least 2/21.
(ii): The total charge of the crossing-vertices is at most $8n - 8$.

PROOF. (i) Let $f_1, \ldots f_4$ be the four faces adjacent to a crossing-vertex, in counterclockwise direction. We have to prove that at least one of them has positive modified charge, that is, for some f_i, $\mathrm{ch}_1(f_i) > 0$. Suppose this is not the case. Then each of $f_1, \ldots f_4$ is either a 0-triangle or a 0-quadrilateral. If two neighboring faces, say f_1 and f_2, are 0-quadrilaterals, then we have a $K_{2,3}$ in $G(\mathcal{S})$ (see Figure 7A). If two neighboring faces, say f_1 and f_2, are 0-triangles, then we have two segments crossing twice. Therefore, up to symmetry, the only remaining case is when f_1 and f_3 are 0-triangles, and f_2 and f_4 are 0-quadrilaterals. However, in this case two segments would again cross twice, which is a contradiction (see Figure 7B).

(ii) Clearly, each face f with $\mathrm{ch}_1(f) > 0$ gives charge to at most $|f|$ crossing-vertices. Since $\mathrm{ch}_1(f) > 2|f|/21$, the total charge of the crossing-vertices is at most as much as the total modified charge of the faces, which is $8n - 8$. □

Lemma 3.3 (i) (and Theorem 3.2 (i)) now follow directly from Claim 3.5. Each crossing-vertex has charge at least 2/21, and their total charge is at most $8n - 8$. Therefore, the total number of crossing vertices is at most $84n - 84$.

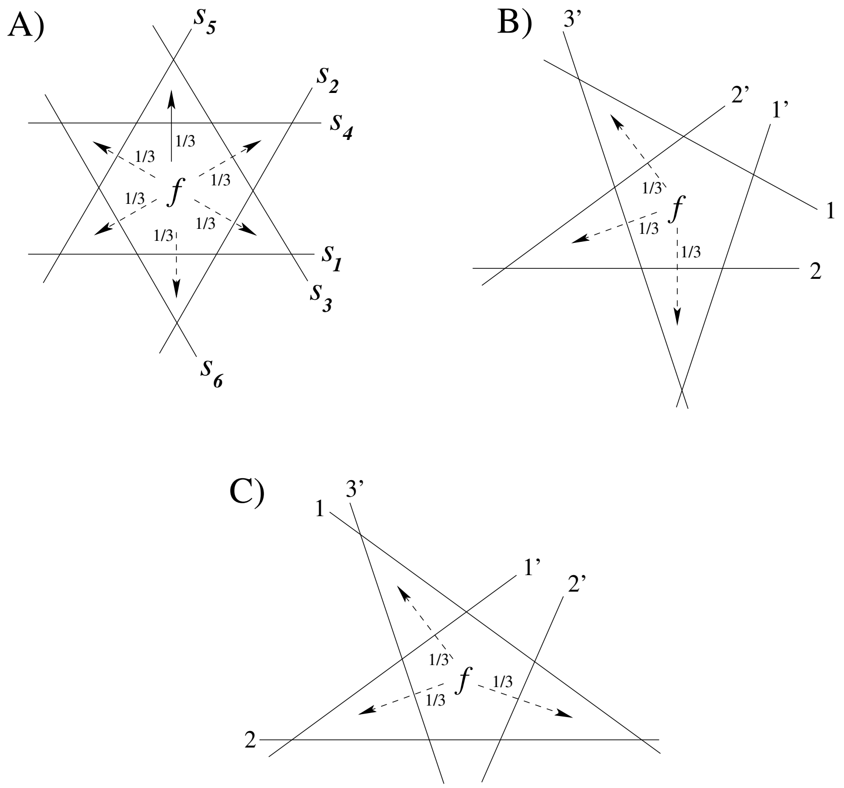

FIGURE 6. Labeled segments $\{1, 2\}$ and $\{1', 2', 3'\}$ form a $K_{2,3}$ in $G(\mathcal{S})$.

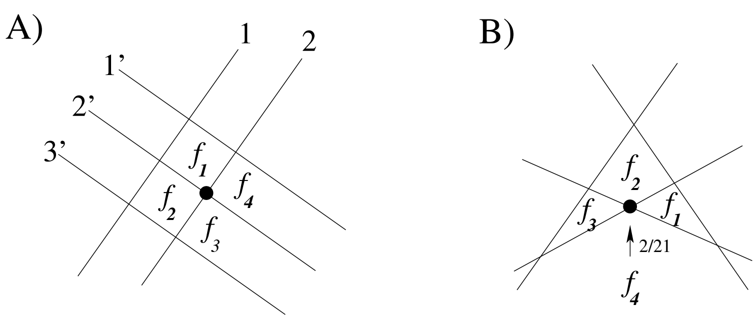

FIGURE 7. A) $\{1, 2\}$ and $\{1', 2', 3'\}$ form a $K_{2,3}$ in $G(\mathcal{S})$; B) If f_2 is a 0-quadrilateral, then $|f_4| > 4$, hence, f_4 discharged $2/21$ to the crossing-vertex.

Case (ii): $G(\mathcal{S})$ does not contain C_6 as a subgraph. We will only sketch the argument, since it is similar to the argument in Case (i). Again, let $\tilde{G}$ be the planar graph together with its planar drawing, induced by $\mathcal{S}$. We use the same notation as in Case (i). A 1-hexagon is a triangle determined by three segments, such that a fourth segment ends inside. For each 1-hexagon, cut the corresponding fourth segment such that it ends just *outside* the triangle (see Figure 8). We have

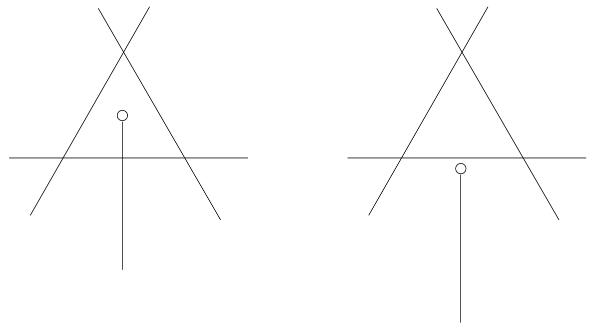

FIGURE 8. Eliminating 1-hexagons.

lost at most $2n$ crossing-vertices and we still do not have a C_6 in the intersection graph. Therefore, it is sufficient to prove the result for the new arrangement. We use the same notation for the new arrangement.

Assign charge $\mathrm{ch}(f) = |f| + v_e(f) - 4$ to each face f. Just like in Case (i), we have

$$\sum_{f \in F(\tilde{G})} \mathrm{ch}(f) = 8n - 8.$$

CHARGE REDISTRIBUTION, STEP 1. This step is identical to CHARGE REDISTRIBUTION in Case (i), so each 0-triangle gets charge $1/3$ through each of its sides from some t-gon, $t \geq 5$. Denote by $\mathrm{ch}_1(f)$ the new charge of a face f and call it a *modified charge*. Now 0-triangles and 0-quadrilaterals have 0 modified charge. 0-pentagons might have modified charge $-2/3$. There are no hexagons at all, since a 0-hexagon would imply a C_6 in the intersection graph, and we eliminated the 1-hexagons.

All other faces have positive charges.

CHARGE REDISTRIBUTION, STEP 2. We redistribute charges again, so that 0-pentagons will have positive charge as well.

Let p be a 0-pentagon which gave charge $1/3$ to five 0-triangles. Clearly, $\mathrm{ch}_1(p) = -2/3$. Let $s_1, \dots, s_5$ be the segments, in counterclockwise direction, which contain the sides of p. It is not hard to see that any two of $s_1, \dots, s_5$ intersect. Therefore, there is no other segment s that intersects two segments from $\{s_1, \dots, s_5\}$, since that would create a C_6 in the intersection graph (see Figure 9A). Moreover, for the same reason, for any $1 \leq i < j \leq 5$, there is no path from s_i to s_j of length 2, 3, or 4 in the intersection graph of $\mathcal{S} \setminus \{s_1, \dots, s_5\} \cup \{s_i, s_j\}$. By these observations, we can conclude that all five neighboring faces of p are 0-triangles (these triangles got the charge $1/3$ from p in CHARGE REDISTRIBUTION, STEP 1). It also follows that all five faces $f_1, \dots, f_5$, sharing a vertex with p but not a side, have $|f_i| \geq 9$ (see Figure 9B-E). Move charge $1/7$ from each f_i to p, and do the same for each 0-pentagon which gave charge $1/3$ to five 0-triangles (see Figure 9F).

Let p be a 0-pentagon which gave charge $1/3$ to four 0-triangles. Let $f_1, \dots, f_5$ be the faces sharing a vertex with p but not a side. By a similar argument as above, one can show that two faces among $f_1, \dots, f_5$ have at least 9 edges along their boundary, and two of them have at least 8 sides. Move charge $1/7$ from each f_i of at least 8 sides to p, and do the same for each such 0-pentagon.

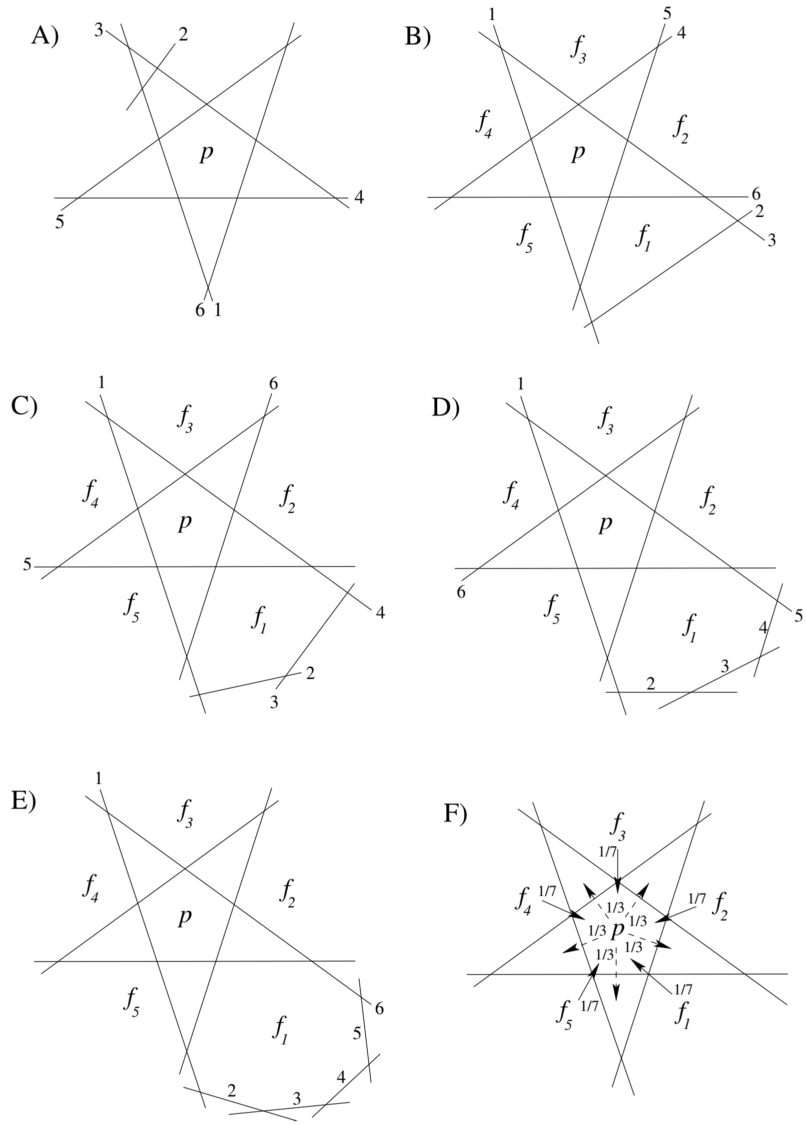

FIGURE 9. Discharging 1/7 through each vertex of p from the neighboring faces f_i, $|f_i| \geq 9$.

Finally, suppose that p is a 0-pentagon which gave charge 1/3 to three 0-triangles. Let $f_1, \ldots, f_5$ be defined as before. Then, again by an analogous argument, we can show that at least three faces among $f_1, \ldots, f_5$ are of size at least 8. Move charge 1/7 from each f_i of at least 8 sides to p, and do the same for each such 0-pentagon.

There are some technicalities that we are omitting in the last two cases, as some of the faces $f_1, \ldots, f_5$ could be identical. This does not affect our charging redistribution or computation later on, but it does increase the number of cases to be considered. Furthermore, some of the faces f_i could potentially have $|v_e(f_i)| > 0$,

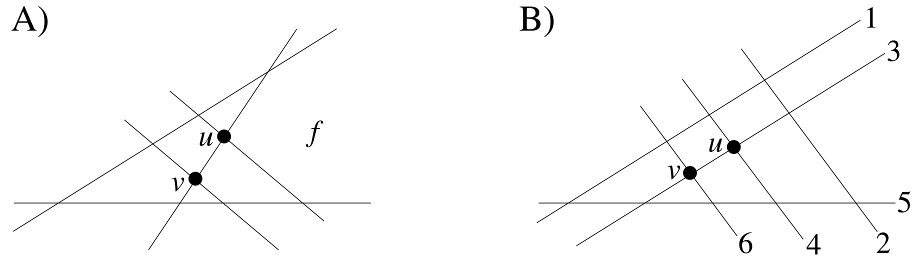

FIGURE 10. A) $|f| > 4$, so $\mathrm{ch}_3(u) \geq 1/105$ and $\mathrm{ch}_4(v) \geq 1/525$;
B) Labeled segments form a C_6 in $G(\mathcal{S})$.

in which case $|f_i| \geq 7$; however, these faces have plenty of charge for discharging to p, and the inequalities in our later computation are even easier to establish.

Denote by $\mathrm{ch}_2(f)$ the new charge of a face f and call it the *final face-charge.*

CLAIM 3.6. We have $\mathrm{ch}_2(f) = 0$ if and only if f is a 0-triangle or a 0-quadrilateral. Otherwise, $\mathrm{ch}_2(f) \geq |f|/105$.

PROOF. It is clear that $\mathrm{ch}_2(f) = 0$ for 0-triangles and 0-quadrilaterals. We show that $\mathrm{ch}_2(f) \geq |f|/105$ for all other types of faces.

Suppose that $|f| \geq 8$. Then an easy calculation shows that $\mathrm{ch}_2(f) \geq |f| - 4 - |f|/3 - |f|/7 = 11|f|/21 - 4 > |f|/105$.

If $|f| = 7$, then $\mathrm{ch}_2(f) = \mathrm{ch}_1(f) \geq 7 - 4 - 7/3 = 2/3 > 7/105$ (the case of 1-heptagons is omitted for the sake of simplicity).

Since there is no C_6 in the intersection graph, and we eliminated all 1-hexagons, $|f| = 6$ is impossible.

Finally, suppose that $|f| = 5$. Then the possible values of $\mathrm{ch}_2(f)$ are 1, 2/3, 1/3, $0 + 3/7$, $-1/3 + 3/7$, and $-2/3 + 5/7$, the smallest being $-2/3 + 5/7 = 1/21$, so we are done. $\square$

Now move the charges from the faces to the crossing-vertices.

CHARGING CROSSINGS, STEP 1. For each face with $\mathrm{ch}_2(f) > 0$, we know that $\mathrm{ch}_2(f) \geq |f|/105$. Move 1/105 charge to each vertex on its boundary. For any vertex v, let $\mathrm{ch}_3(v)$ denote the charge of v.

CHARGING CROSSINGS, STEP 2. For any vertex v, let

$$\mathrm{ch}_4(v) = \frac{1}{5} \sum_{\substack{u = v, \text{ or} \\ uv \in E(\tilde{G})}} \mathrm{ch}_3(u).$$

CLAIM 3.7. **(i):** Each crossing vertex gets charge at least 1/525.
(ii): The total charge of the crossing-vertices is at most $8n - 8$.

PROOF. (i) Suppose for contradiction that $\mathrm{ch}_4(v) < 1/525$. Then $\mathrm{ch}_3(u) = 0$ if $u = v$ or u is a neighbor of v in $\tilde{G}$; consequently, all faces adjacent to v, and all its neighbors have charge 0, so all these faces are 0-triangles or 0-quadrilaterals. In particular, v and all its neighbors are crossing-vertices. However, a straightforward case analysis shows that in this case we have a C_6 in the intersection graph of the corresponding segments. Two cases are depicted in Figure 10.

(ii) Clearly, each face f with $\mathrm{ch}_2(f) > 0$ gives charge to at most $|f|$ vertices. Since $\mathrm{ch}_2(f) \geq |f|/105$, after the first step, the total charge of the vertices is at most as much as the total final face-charge of the faces, which is $8n-8$. After the second step, the total charge of the vertices could not increase. More precisely,

$$\sum_{v \in V_c(\tilde{G})} \mathrm{ch}_4(v) \le \sum_{v \in V(\tilde{G})} \mathrm{ch}_4(v) \le \sum_{v \in V(\tilde{G})} \mathrm{ch}_3(v) \le \sum_{f \in F(\tilde{G})} \mathrm{ch}_2(f) = 8n-8.$$

□

Now we are in a position to conclude our proof of Lemma 3.3 (ii) (and, thus, Theorem 3.2 (ii)). Each crossing-vertex has charge at least $1/525$, and their total charge is at most $8n-8$. Therefore, the total number of crossing-vertices is at most $4200n-4200$. □

Concluding remarks. Using the same method, we can prove Conjecture 3.1 in the case when the forbidden subgraph H is either C_8 or $K_{2,4}$. However, since the discharging method is always based on local discharging rules and arguments, as the size of H increases, our proofs get increasingly complicated.

On the other hand, in a forthcoming paper, Fox and Pach [**FP**] settle Conjecture 3.1 using completely different techniques.

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