2. The Tangent Line

Tangent Line

The tangent line to a circle at a point *P* on its circumference is the line perpendicular to the radius of the circle at *P*. In Figure 1, The line *T* is the tangent line which is perpendicular to the radius of the circle at the point *P*.

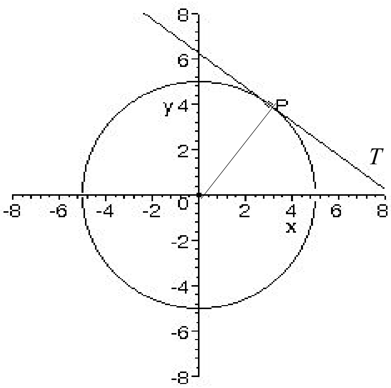

Figure 1: A Tangent Line to a Circle

While the tangent line to a circle has the property that it is perpendicular to the radius at the point of tangency, it is not this property which generalizes to other curves. We shall make an observation about the tangent line to the circle which is carried over to other curves, and may be used as its defining property.

Let us look at a specific example. Let the equation of the circle in Figure 1 be $x^2 + y^2 = 25$. We may easily determine the equation of the tangent line to this circle at the point *P*(3,4). First, we observe that the radius is a segment of the line passing through the origin (0, 0) and *P*(3,4), and its equation is $y = \frac{4}{3}x$ (why?). Since the tangent line is perpendicular to this line, its slope is -3/4 and passes through *P*(3, 4), using the point-slope formula, its equation is found to be

$$y = -\frac{3}{4}x + \frac{25}{4}$$

Let us compute y-values on both the tangent line and the circle for *x*-values *near* the point *P*(3, 4). Note that near *P*, we can solve for the *y*-value on the upper half of the circle which is found to be

$$y = \sqrt{25 - x^2}$$

When $x = 3.01$, we find the *y*-value on the tangent line is $y = -3/4(3.01) + 25/4 = 3.9925$, while the corresponding value on the circle is $y = \sqrt{25 - (3.01)^2} \approx 3.99248$. (Note that the tangent line lie above the circle, so its *y*-value was expected to be a larger.) In Table 1, we indicate other corresponding values as we vary *x near P*. Construction of Table 1 can be done easily on your calculator by setting $y_1(x) = -\frac{3}{4}x + \frac{25}{4}$ and $y_2(x) = \sqrt{25 - x^2}$, and using the Table feature.

Note the closer we get to the point *P*, the better the *y*-values agree, and near the point of tangency, the *y*-values on the circle and line are nearly the same. This observation motivates an alternative definition of a tangent line to a smooth curve at a point *P*.

x	y1	y2			
2.96	4.03	4.0297			
2.97	4.0225	4.0223			
2.98	4.015	4.0149			
2.99	4.0075	4.0075			
3.	4.	4.			
3.01	3.9925	3.9925			
3.02	3.985	3.9849			
3.03	3.9775	3.9773			

x=3.

MAIN RAD AUTO FUNC

Table 1: Comparing the y-values near $P(3, 4)$ on $y = -\frac{3}{4}x + \frac{25}{4}$ and $x^2 + y^2 = 25$

DEFINITION

The tangent line to a smooth curve at a point P is the best linear approximation to the curve at that point. That is, among all different lines that touch the curve at the point P, the tangent line is the one whose y-values best approximate the y-values on the curve near the point of tangency.

Notice the definition says nothing about the tangent line being perpendicular to other lines at the point of tangency, as in the case of the circle where it is perpendicular to the radius. The notion of perpendicularity is unique to the circle. (The term *smooth* used in the definition will be clarified when you study calculus, where you will discover that some curves do not have tangent lines at points which are "sharp.")

As an illustration, we indicate in, Figure 2,the tangent line to the graph of $y = x^3$ at the point (2,3). Observe how well the tangent line approximates the curve near the point, that is, the y-values are almost identical. The determination of the equation of the tangent line is studied in calculus.

The observation that a tangent line approximates the curve near the point of tangency is an extremely useful tool, one that is used over and over again in mathematics and is sometimes called *linearization*.

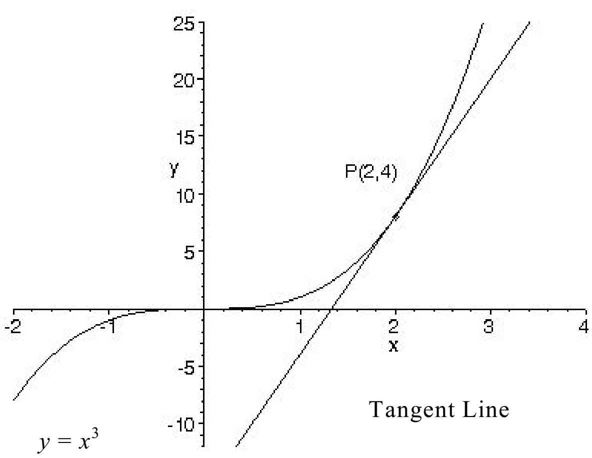

Figure 2: The Tangent line to $y = x^3$ at the Point $P(2, 4)$

Exercise Set 2

1. (a) Determine the equation of the tangent line to the circle $x^2 + y^2 = 169$ at the point (5, 12). (b) Compare the y-values on the tangent line with those on the circle *near* $x = 5$.

2. (a) Determine the equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point (3, -4). (b) Compare the y-values on the tangent line with those on the circle *near* $x = 3$.

3. (a) Determine the equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point (4, 3). (b) Compare the y-values on the tangent line with those on the circle *near* $x = 4$.

4. (a) Determine the equation of the tangent line to the circle $x^2 + y^2 = 169$ at the point (12, -5). (b) Compare the y-values on the tangent line with those on the circle *near* $x = 12$.

(Notes)