CHEBYSHEV (TCHEBYCHEFF)

On the Totality of Primes

(Translated from the French by Professor J. D. Tamarkin, Brown University, Providence, Rhode Island.)

Pafnuty Lvovich Chebyshev (Tchebycheff, Tchebytcheff) was born on May 14, 1821, and died on Nov. 26, 1894. He is one of the most prominent representatives of the Russian mathematical school. He made numerous important contributions to the theory of numbers, algebra, the theory of probabilities, analysis, and applied mathematics. Among the most important of his papers are the two memoirs of which portions are here translated:

1. "Sur la totalité des nombres premiers inférieurs à une limite donnée," *Mémoires presentés à l'Académie Impériale des Sciences de St.-Pétersbourg par divers savants et lus dans ses assemblées,* Vol. 6, pp. 141–157, 1851 (Lu le 24 Mai, 1848); *Journal de Mathématiques pures et appliquées,* (1) Vol. 17, pp. 341–365, 1852; *Oeuvres,* Vol. 1, pp. 29–48, 1899.

2. "Mémoire sur les nombres premiers," *ibid.*, Vol. 7, pp. 15–33, 1854 (lu le 9 Septembre, 1850), *ibid.*, pp. 366–390, *ibid.*, pp. 51–70.

These memoirs represent the first definite progress after Euclid in the investigation of the function $\phi(x)$ which determines the totality of prime numbers less than the given limit x. The problem of finding an asymptotic expression for $\phi(x)$ for large values of x attracted the attention and efforts of some of the most brilliant mathematicians such as Legendre, Gauss, Lejeune-Dirichlet, and Riemann.

Gauss (1791, at the age of fourteen) was the first to suggest, in a purely empirical way, the asymptotic formula $\frac{x}{\log x}$ for $\phi(x)$. (*Werke,* Vol. X_1, p. 11, 1917.) Later on (1792–1793, 1849), he suggested another formula $\int_2^x \frac{dx}{\log x}$, of which $\frac{x}{\log x}$ is the leading term (Gauss's letter to Encke, 1849, *Werke,* Vol. II, pp. 444–447, 1876). Legendre, being, of course, unaware of Gauss's results, suggested another empirical formula $\frac{x}{A \log x + B}$ (*Essai sur la théorie des nombres,* 1st ed., pp. 18–19, 1798) and specified the constants A and B as $A = 1$, $B = -1.08366$ in the second edition of the *Essai* (pp. 394–395, 1808). Legendre's formula, which Abel quoted as "the most marvelous in mathematics" (letter to Holmboe, *Abel Memorial,* 1902, Correspondence, p. 5), is correct up to the leading term only. This fact was recognized by Dirichlet ("Sur l'usage des séries infinies dans la théorie des nombres," *Crelle's Journal,* Vol. 18, p. 272, 1838, in his remark written on the copy presented to Gauss. *Cf.* Dirichlet, *Werke,* Vol. 1, p. 372, 1889). In this note to Gauss, Dirichlet suggested another formula $\sum^{x} \frac{1}{\log n}$. The proof of these results, although announced by Dirichlet, has never been published, so that Chebyshev's (Tchebycheff's) memoirs should be considered as the first attempt at a rigorous investigation of the problem by analytical methods. Chebyshev did not reach the final goal—to prove that the ratio $\phi(x) : \frac{x}{\log x}$ tends to 1 as $x \rightarrow \infty$. This important theorem was proved some 40 years later by Hadamard ("Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques," *Bulletin de la Societé Mathématique de France*, Vol. 24, pp. 199–220, 1896) and by de la Vallée Poussin ("Recherches analytiques sur la théorie des nombres premiers," *Annales de la Société Scientifique de Bruxelles*, Vol. 20, pp. 183–256, 1896), their work being based upon new ideas and suggestions introduced by Riemann ("Über die Anzahl der Primzahlen unter einer gegebenen Grenze," *Monatsberichte der Berliner Akademie*, pp. 671–680, 1859; *Werke*, 2nd ed., pp. 145–153, 1892).

Although Chebyshev did not prove this final theorem, still he succeeded in obtaining important inequalities for the function $\phi(x)$, which enabled him to investigate the possible forms of approximation of $\phi(x)$ by means of expressions containing algebraically x, e^x, $\log x$ (*Memoir* 1, above) with a conclusion concerning the rather limited range of applicability of Legendre's formula. In the *Memoir* 2, Chebyshev obtains rather narrow limits for the ratio $\phi(x) : \frac{x}{\log x}$, which provide a proof for the famous Bertrand postulate: "If $x \geqq 2$, there is at least one prime number between x and $2x - 2$."

Memoir 1: On the Function which Determines the Totality of Primes Less than a Given Limit

§1. Legendre in his *Théorie des nombres*[1] proposes a formula for the number of primes between 1 and any given limit. He begins by comparing his formula with the result of counting the primes in the most extended tables, namely those from 10,000 up to 1,000,000, after which he applies his formula to the solution of many problems. Later the same formula has been the object of investigations of Mr. Lejeune-Dirichlet who announced in one of his memoirs in *Crelle's Journal*, Vol. 18, that he had found a rigorous analytical proof of the formula in question.[2] Despite the authority of the name of Mr. Lejeune-Dirichlet and the pronounced agreement of the formula of Legendre with the tables of primes we permit ourselves to raise certain doubts as to its

[1] Volume 2, p. 65 (3rd edition).

[2] [Naturally Chebyshev was unaware of the marginal notation made by Dirichlet in the copy of his paper presented to Gauss, to which we referred above.]